This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

Math 5009.00

BOOK PLATE OF
JOHN QUINCY ADAMS
1831

THE LIBRARY OF
ADAMS HOUSE
HARVARD UNIVERSITY
1931

TRANSFERRED
TO
[redacted]

FAMOUS GEOMETRICAL THEOREMS AND PROBLEMS

WITH THEIR HISTORY

BY

WILLIAM W. RUPERT, C.E.

SUPERINTENDENT OF SCHOOLS, POTTSTOWN, PA.

> "I am sure that no subject loses more than mathematics by any attempt to dissociate it from its history." — J. W. L. Glaisher.

BOSTON, U.S.A.
D. C. HEATH & CO., PUBLISHERS
1900

Math 5009.00

~~Math~~
~~452~~

HARVARD
UNIVERSITY
LIBRARY
MAR 30 1956

COPYRIGHT, 1900
BY D. C. HEATH & CO.

PREFACE.

THE author, having derived much pleasure and inspiration from the brief historical notes in some of the mathematical text-books that he studied when a student in college, has thought that, by giving the history of a few of the most celebrated geometrical theorems and problems, he might place a "light in the window" which may throw a cheerful ray adown the long and sometimes dusty pathway that leads to geometrical truth.

In the preparation of this little book most valuable assistance has been derived from Florian Cajori's *History of Mathematics*, James Gow's *History of Greek Mathematics*, and G. J. Allman's *Greek Geometry from Thales to Euclid.* It is, however, to W. W. Rourse Ball's remarkably interesting *Short History of Mathematics* that *Famous Geometrical Theorems and Problems* owes the largest debt.

To Professor A. D. Eisenhower, Principal of the Norristown High School, George Q. Sheppard, Professor of Mathematics, Hill School, Pottstown, Pa., Dr. George M. Philips, Principal West Chester State Normal School, and Daniel Carhart, C.E., Dean and Professor of Civil Engineering, Western University of Pennsylvania, who have read this book in manuscript, the author is indebted for valuable suggestions and many kind words of encouragement.

For the excellent and accurate diagrams, and for proof No. V. in Chapter I., my thanks are due to my young friend and former pupil, Luther D. Showalter, C.E.

WILLIAM W. RUPERT.

POTTSTOWN, PA.,
December 23, 1899.

WELLS'S MATHEMATICS

ESSENTIALS OF ALGEBRA
ACADEMIC ALGEBRA
NEW HIGHER ALGEBRA
HIGHER ALGEBRA
UNIVERSITY ALGEBRA
COLLEGE ALGEBRA

ESSENTIALS OF GEOMETRY (1899) Plane and Solid
ELEMENTS OF GEOMETRY (1894) Plane and Solid

NEW PLANE AND SPHERICAL TRIGONOMETRY
PLANE TRIGONOMETRY
ESSENTIALS OF TRIGONOMETRY

NEW SIX PLACE LOGARITHMIC TABLES
FOUR PLACE TABLES

ACADEMIC ARITHMETIC

D. C. HEATH & CO., Publishers
BOSTON NEW YORK CHICAGO LONDON

FAMOUS GEOMETRICAL THEOREMS.

CHAPTER I.

THEOREM.

1. *The sum of the three angles of every plane triangle is equal to two right angles.*

The mathematical truth enunciated in the above theorem is not new. It has been known for more than two thousand years. Thales, one of the seven sages of Greece, who was born about 640 B.C., must have been aware that the sum of the angles of a triangle is equal to two right angles.

Our reason for believing that Thales was not ignorant of the theorem under consideration is found in the beautiful demonstration by which he proved that every angle in a semicircle is a right angle. This appears to have been regarded as the most remarkable of the geometrical achievements of Thales, and it is stated that on inscribing a right-angled triangle in a circle he sacrificed an ox to the immortal gods.

Before giving the proof by which Thales probably established the truth that every angle inscribed in a semicircle is a right angle, it will be well to consider the geometrical capital which this Grecian mathematician had at his command.

The following, says Professor W. W. R. Ball in his *History of Mathematics*, comprise all the propositions that we can now with reasonable probability refer back to him:

i. *The angles at the base of an isosceles triangle are equal.*

ii. *If two straight lines cut one another, the vertically opposite angles are equal.*

Thales may have regarded this as obvious. Euclid, who was of Greek descent, and who was born about 330 B.C., was the first to give a strict proof of it.

iii. *A triangle is determined if its base and base angles are given.*

Apparently this was applied to find the distance of a ship at sea; the base probably being a tower, and the base angles being obtained by observation.

iv. *The sides of equiangular triangles are proportional.*

This is said to have been used by Thales in Egypt to find the height of a pyramid. In a dialogue given by Plutarch, the speaker, addressing Thales, says, "Placing your stick at the end of the shadow of the pyramid, you made by the sun's rays two triangles, and so proved that the [height of the] pyramid was to the [length of the] stick as the shadow of the pyramid to the shadow of the stick." The king, Amasis, who was present, is said to have been amazed at this application of abstract science, and the Egyptians seem to have been previously unacquainted with the theorem.

v. *A circle is bisected by any diameter.*

This may have been enunciated by Thales, but it must have been recognized as an obvious truth from the earliest times.

vi. *The angle in a semicircle is a right angle.*

It is believed that Thales proved this proposition in the following manner:

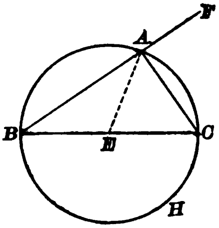


Fig. 1.

Let *ABCH* be a circle of which the diameter is *BC*, and the centre *E*. Prove that the angle *BAC* inscribed in the semicircle is a right angle.

Draw *AE* and produce *BA* to *F*.

Because *BE* is equal to *EA*, the angle *EAB* is equal to *EBA*; also, because *AE* is equal to *EC*, the angle *EAC* is equal to *ECA*; wherefore, the whole angle *BAC* is equal to the two angles *ABC*, *ACB*.

But *FAC*, the exterior angle of the triangle *ABC*, is also equal to the two angles *ABC*, *ACB*; therefore the angle *BAC* is equal to the angle *FAC*, and each of them is therefore a right angle; wherefore, the angle *BAC* in a semicircle is a right angle.

Thales's demonstration, if we may call this his, is quite different from the one given in modern text-books; but it is certainly neither less rigid nor less beautiful. The demonstration is the one given in Euclid, but his work, we must remember, is, to a large extent, compiled from the works of previous writers. It will be seen, however, that this demonstration implies a knowledge of a seventh proposition, — "If one side of a triangle be produced, the exterior angle is equal to the sum of the two interior and opposite angles." Thales must have been familiar with this truth.

2. Below we give a brief and beautiful proof of the truth that the angle in a semicircle is a right angle; but this, too, implies a knowledge of the proposition quoted above:

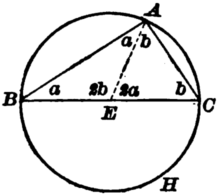


Fig. 2.

BAC is the angle inscribed in a semicircle.

The exterior angle $BEA = 2b$.

The exterior angle $AEC = 2a$.

But $2a + 2b = 2$ right angles.

$\therefore a + b$ or angle $BAC =$ a right angle.

3. The following demonstration might have been given; but it rests directly upon the fact that the sum of the three angles of a triangle is equal to two right angles, while Thales's proof, and the one just given, lead up to this knowledge:

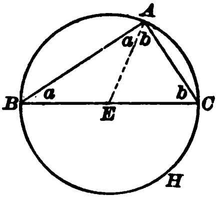


Fig. 3.

BAC is the angle inscribed in a semicircle.

Sum of angles of two triangles ABE and AEC $= 4$ right angles.

Angles about E on one side of $BC = 2$ right angles.

Remaining angles of two triangles $= 2$ right angles.

$$\therefore\ a + b + a + b = 2 \text{ right angles,}$$

or

$$2(a + b) = 2 \text{ right angles.}$$

$\therefore\ a + b$, or angle BAC inscribed in a semicircle, is a right angle.

It has been ingeniously suggested that the shape of the tiles used in paving floors may have afforded an experimental demonstration of the fact that the sum of the angles of a triangle is equal to two right angles. We know from Eudemus, who was a pupil of Aristotle, that the first geometricians proved the general property for three species of triangles, and it is highly probable that the proof was of this experimental character. The area about a point can be filled by the angles of six equilateral triangles or tiles, hence the proposition is true for an equilateral triangle. Again, a rectangle (the sum of whose angles is four right angles) can be divided into two equal right-angled triangles, hence the proposition is true for a right-angled triangle. It is plain that tiles of such a shape would give an ocular demonstration of this case.

4. Lastly, any triangle can be divided into two right-angled triangles by letting fall a perpendicular from the largest angle upon the opposite side, and therefore again the proposition is true.

This truth, in the last case, becomes apparent from an examination of the following diagram:

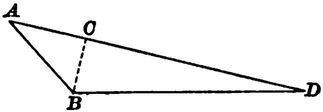


Fig. 4.

The perpendicular from *B* divides the triangle *ABD* into two right-angled triangles, *ABC* and *BCD*. But it has been shown before that the angles of a right-angled triangle are together equal to two right angles, hence the angles of *ABC* and $BCD = 4$ right angles. But the angles about *C* on one side of $AD = 2$ right angles, hence the angles of the triangle $ABD = 2$ right angles.

It is believed that the demonstration for the right-angled triangle was at first worked out for isosceles right-angled triangles only, but it was subsequently extended so as to cover any right-angled triangle.

From all this it will be seen that the Greek geometricians were timid about generalizing their proofs, and were afraid that any additional condition imposed on the triangle might vitiate the general result.

We now desire to direct attention to two other mechanical proofs of the theorem which forms the subject of this chapter. The first was discovered by Blaise Pascal when a mere boy of about twelve years, and the other is suggested by Professor Ball in his *History of Mathematics*.

Blaise Pascal was born at Clermont, in Auvergne, on June 19, 1623. In 1631 his father moved to Paris, partly to prosecute his own scientific studies, partly to educate his only son, who had already exhibited remarkable ability.

Pascal was kept at home that he might not be overworked, and with the same object in view, for the boy was delicate, it was decided that his education should at first be confined to Latin and Greek. All mathematical books were hidden out of his sight. The boy once asked his father, who was a mathematician of more than ordinary ability, what mathematics treated of, and was answered in general "that it was the method of making figures with exactness, and of finding out what proportions they relatively had to one another." He was, however, at the same time forbidden to talk any more about the subject. Stimulated, perhaps, by the injunction against reading mathematics, Pascal gave up his play-time to this new study, and in a few weeks had discovered, without aid from any one, many important properties of figures, and in particular the proposition that the sum of the angles of a triangle is equal to two right angles. Pascal made this discovery by turning the angular points of a triangular piece of paper over so as to meet in the centre of the inscribed circle. Cajori, in his *History of Mathematics*, says, "His father caught him in the act of studying the theorem, and was so astonished at the sublimity and force of his genius as to weep for joy."

At the age of fourteen, Pascal was admitted to the weekly meetings of Roberval, Mersenne, Mydorge, and other famous French geometricians. At sixteen he wrote an essay on conic sections, which at the time was considered superior to anything that had appeared since the time of Archimedes,—a period of more than eighteen hundred years. Descartes would not at first believe that Pascal's conics had been written by one so young. Unfortunately this treatise was never published, and is now lost. Its author died from overwork — the very thing from which his father tried to shield him—at the early age of thirty-nine.

Professor Ball's suggestion is that a demonstration similar to Pascal's may be obtained by turning the angular points of a paper triangle over so as to meet at the foot of the perpendicular drawn from the largest angle to the opposite side.

5. We now give five proofs of the proposition enunciated at the beginning of the chapter:

PROOF NUMBER I.

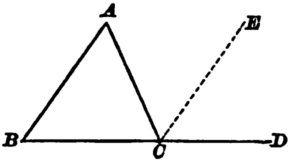


Fig. 5.

Let ABC be a triangle, and let one of its sides BC be produced to D; the exterior angle ACD is equal to the two interior and opposite angles CAB, ABC; and the three interior angles of the triangle, viz. ABC, BCA, CAB, are together equal to two right angles.

Through the point C draw CE parallel to the straight line AB; and because AB is parallel to CE, and AC meets them, the alternate angles BAC, ACE are equal. Again, because AB is parallel to CE, and BD falls upon them, the exterior angle ECD is equal to the interior and opposite angle ABC, but the angle ACE was shown to be equal to BAC; therefore the whole exterior angle ACD is equal to the two interior and opposite angles

CAB, *ABC*; to these angles add the angle *ACB*, and the angles *ACD*, *ACB* are equal to the three angles *CBA*, *BAC*, *ACB*; but the angles *ACD*, *ACB* are equal to two right angles; therefore also the angles *CBA*, *BAC*, *ACB* are equal to two right angles.

This is the demonstration, word for word, found in Euclid's Elements. However, it was Pythagoras, who was born about 569 B.C., and not Euclid, who worked out this particular proof. If this old Grecian mathematician could come back to earth at the close of the nineteenth century, he would behold some astounding changes; but he could still find his own beautiful proof of the proposition under consideration in nearly every modern textbook on geometry.

Euclid was a very successful teacher and was the author of several works, but his reputation has rested mainly on his Elements. This work contains a systematic exposition of the leading propositions of elementary geometry. Euclid is said to have insisted that knowledge was worth acquiring for its own sake, and we are told that when a lad who had just begun geometry asked, "What do I gain by learning all this stuff?" Euclid made his slave give the boy some coppers, "Since," said he, "he must make a profit out of what he learns." The same question has, doubtless, been asked more than once since Euclid's day, but it is not probable the same answer has been given.

6. PROOF NUMBER II.

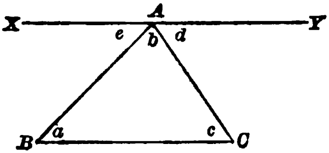


Fig. 6.

Through vertex *A* draw line *xy* parallel to side *BC*.

Then $a = e,\ c = d,\ b = b.$

They are alternate interior angles.

$$\therefore\ a + c + b = e + d + b.$$

But $e + d + b = 2$ right angles.

$$\therefore\ a + c + b = 2 \text{ right angles.}$$

7. PROOF NUMBER III.

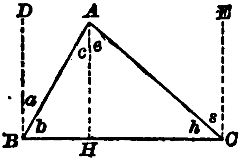


Fig. 7.

Draw lines *DB*, *AH*, *EC* perpendicular to *BC*. Since these lines are perpendicular to the same line, they are parallel.

Then $a = c$, $s = e$, $b + h = b + h$.

They are alternate interior angles.

$$\therefore\ a + s + b + h = c + e + b + h.$$

But $a + s + b + h = 2$ right angles.

$\therefore\ c + e + b + h$ or $BAC + ABC + ACB = 2$ right angles.

8. PROOF NUMBER IV.

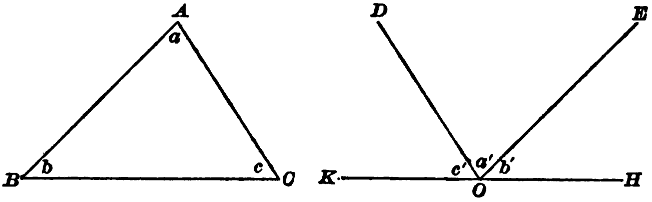


Fig. 8.

Draw KH parallel to BC, and from O, any point of this line, draw OE and OD parallel respectively to the sides AB and AC.

Then $a = a'$, $b = b'$, $c = c'$.

(Having their sides parallel, right to right, and left to left.)

$$\therefore\ a + b + c = a' + b' + c'.$$

But $a' + b' + c' = 2$ right angles.

$$\therefore\ a + b + c = 2 \text{ right angles}.$$

This is the proof given in Phillips and Fisher's geometry.

9. PROOF NUMBER V.

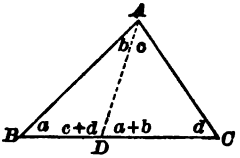


Fig. 9.

From any vertex A, draw to any point in the side opposite, D, a line AD.

Then the exterior angle $ADC = a + b$.

Also, the exterior angle $BDA = c + d$.

Whence $a + b + c + d = 2$ right angles.

$\therefore B + A + C = 2$ right angles.

CHAPTER II.

THEOREM.

10. *The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides.*

Pythagoras, who was born at Samos about 569 B.C., was the first man to prove this important and famous theorem. He, however, had probably learned from the Egyptians the truth of the theorem in the special case when the sides are 3, 4, 5, respectively.

Two thousand years before Christ the Egyptians constructed right angles by so placing three pegs that a cord measured off into 3, 4, and 5 units would just reach around them. The Egyptians, as we know, were very particular about the exact orientation of their temples; and, that there might be no mistake in this all-important matter, they spared no pains to obtain with accuracy a north and south line, and also an east and west line. By carefully noting the points on the horizon where a star rose and set, and then running a line midway between these points, they obtained the desired north and south line.

But how did they get the east and west line, which had to be drawn at right angles to this? For this purpose, professional "rope-fasteners" were employed, who stretched a rope around three pegs (the two of them which were nearest together being placed along the north and south line already determined) so that the sides of the triangle formed were in the ratio of the numbers 3, 4, and 5. The angle opposite the longest side would then be a right angle; and while one leg of the triangle ran north and south, the other leg ran east and west.

Many carpenters and masons, even at the present time, make right angles by measuring off 6 feet and 8 feet in such a manner that a "ten-foot pole" completes the triangle; and, indeed, in the absence of surveying instruments, this is a very expeditious and satisfactory method by which to place lines or foundation walls at right angles.

Pythagoras was not only a great mathematician for his day, but he was an able and enthusiastic teacher. The schools which he opened were crowded with eager, earnest students. Even the women flocked to hear him, though in doing so they broke a law of that time which forbade their going to public meetings.

Amongst his most attentive auditors was Theano, the beautiful and accomplished daughter of his host Milo, whom he married. Theano wrote a biography of her distinguished husband; but, most unfortunately, it is lost.

The followers of Pythagoras formed a brotherhood with all things in common. Their food was simple; their discipline severe; and their mode of life arranged to encourage self-control, temperance, purity, and obedience.

"They rose," says Professor Ball, "before the sun, and began by recalling the events of the preceding day; next they made a plan for the day then commencing, and finally on retiring to rest they were expected to compare their performances with this plan."

Pythagoras did not allow the use of text-books, and all the knowledge of his school was kept from the outside world. The brotherhood of Pythagoras was, in short, a secret society, and for that reason became an object of suspicion to those who did not belong to it. Finally the opposition terminated in violent hostility, and Pythagoras himself was murdered in a popular outbreak in 500 B.C.

Proof Number I.

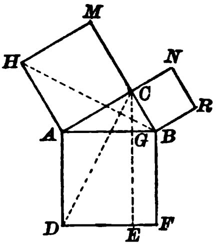


Fig. 10.

The proof that accompanies the above diagram is familiar to every schoolboy, hence we shall not take the space to reproduce it. This proof, found in Euclid's Elements, and in every modern text-book on geometry, was of Euclid's own invention. It is, therefore, more than two thousand years old. Not a little curiosity has been excited to discover what demonstration Pythagoras gave of this famous theorem. "It would seem most likely," says Professor Ball, "to have been one of the two following."

11. Proof Number II.

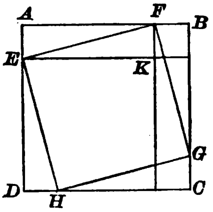


Fig. 11.

Any square *ABCD* can be split up into two squares *BK* and *DK*, and two equal rectangles *AK* and *CK*; that is, it is equal to the square on *FK*, the square on *EK*, and four times the triangle *AEF*. But, if points be taken, *G* on *BC*, *H* on *CD*, and *E* on *DA*, so that *BG*, *CH*, and *DE* are each equal to *AF*, it can easily be shown that *EFGH* is a square, and that the triangles *AEF*, *BFG*, *CGH*, and *DHE* are equal: thus the square *ABCD* is also equal to the square on *EF*, and four times the triangle *AEF*.

$$\therefore \overline{EF}^2 = \overline{FK}^2 + \overline{EK}^2$$

12. Proof Number III.

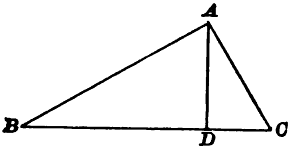


Fig. 12.

Let *ABC* be a right-angled triangle, *BAC* being the right angle. Draw *AD* perpendicular to *BC*. The triangles *ABC* and *DBA* are similar.

$$\therefore BC : AB :: AB : BD.$$

Similarly, $BC : AC :: AC : DC.$

Hence, $\overline{AB}^2 + \overline{AC}^2 = BC(BD + DC) = \overline{BC}^2.$

This proof, of course, involves a knowledge of several important theorems; but with all of these Pythagoras was perfectly familiar.

The following demonstration was discovered by Bhaskara, a Hindoo mathematician, who was born in 1114 A.D. Cajori gives the proof in his *History of Mathematics.*

13. PROOF NUMBER IV.

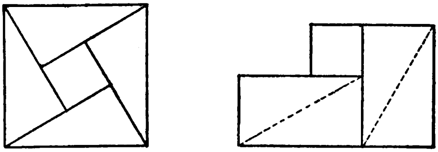


Fig. 13.

Bhaskara draws a right triangle four times in the square of the hypotenuse, so that in the middle there remains a square whose side equals the difference between the two sides of the right triangle. Arranging this square and the four triangles in a different way (Fig. 2), they are seen, together, to make up the sum of the squares of the two sides. "Behold!" says Bhaskara, without giving another word of explanation.

Bhaskara also gives a second demonstration, which, singularly enough, is the same as Number III. above. Pythagoras and Bhaskara, of course, discovered this proof independently; but it was unknown in Europe till John Wallis, an

Englishman, rediscovered it about five hundred years after Bhaskara's day.

For the next nineteen demonstrations, I am indebted to Dr. A. E. Winship, editor of the *New England Journal of Education*, who, with that courtesy for which he is distinguished, has given me permission to reproduce them in this work. A number of the demonstrations appeared in the mathematical department of the *Journal*, edited by Dr. F. P. Matz, several years ago. Others were collected by R. Fleming, C.E., of Lafayette College, and appeared at different times in the *Journal of Education*.

14. Proof Number V.

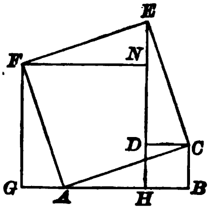


Fig. 14.

ABC is the right-angled triangle. The four triangles ABC, AGF, FEN, EDC are equal to each other.

$HNFG$ is a square, and is equal to the square on AB.

The whole figure $GBCEF =$ squares on AB and $BC + 2$ of four equal triangles.

$$\therefore \overline{AC}^2 = \overline{BC}^2 + \overline{AB}^2.$$

15. PROOF NUMBER VI.

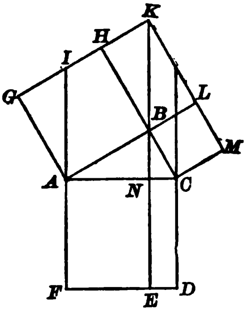


Fig. 15.

ABC is the right-angled triangle with the right angle at *B*.

The square *AGHB* = parallelogram *AIKB* and hence to *ANEF*.

In like manner square *CBLM* = *NCDE*.

$$\therefore \overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2.$$

16. PROOF NUMBER VII.

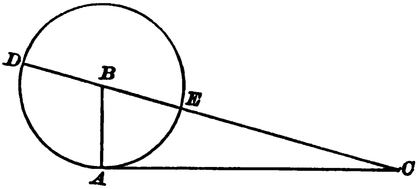


Fig. 16.

$$EC : AC :: AC : DC.$$

But $DC = BC + AB$, $EC = BC - AB$.

$$\therefore\ BC - AB : AC :: AC : BC + AB.$$

$$\overline{BC}^2 - \overline{AB}^2 = \overline{AC}^2, \text{ or } \overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2.$$

17. PROOF NUMBER VIII.

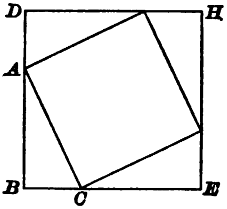


Fig. 17.

In triangle *ABC* produce *BA* to *D*, making $AD = BC$; prolong *BC* to *E*, making $CE = AB$, and complete the square. Erect a square on *AC*.

Then $(AB + AD)^2 =$ area of square *BEHD*. But this area is composed of the area of the four triangles, which are equal to each other, and the square of *AC*; hence,

$$\frac{(AB \times AD)}{2} \times 4 + \overline{AC}^2 = \text{square } BEHD.$$

$$\text{Then } \overline{DB}^2 = \overline{AB}^2 + 2(AB \times AD) + \overline{AD}^2$$
$$= 2(AB \times AD) + \overline{AC}^2.$$

$$\therefore\ \overline{AC}^2 = \overline{AB}^2 + \overline{AD}^2.$$

But $AD = BC.$

$$\therefore\ \overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2.$$

18. PROOF NUMBER IX.

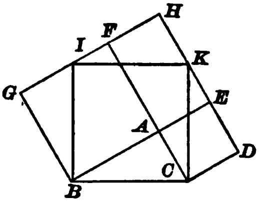


Fig. 18.

BAC is the right-angled triangle, with the right angle at A.

Produce GF, DE till they meet at H. Let BI, CK be each perpendicular to BC and join IK.

The three triangles BGI, IHK, KDC are equal to each other, and to the triangle ABC. Hence the square on $BC = BGHDC - 3$ times the area of triangle ABC.

Now the rectangle $AH = AF \times AE = 2$ times area of triangle ABC; and whole space $BGHDC$ $-$(rectangle $AH +$ triangle $ABC) = \overline{AB}^2 + \overline{AC}^2$. That is, $BGHDC - 3$ times area triangle ABC $= \overline{AB}^2 + \overline{AC}^2$.

$$\therefore \overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2.$$

19. Proof Number X.

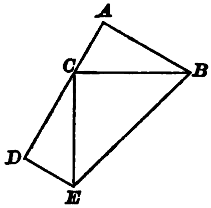


Fig. 19.

On the hypotenuse *CB* of the right-angled triangle *ABC*, draw the half square *CBE*. From *E* let fall the perpendicular *ED* upon the side *AC* produced. The triangles *ABC* and *DCE* are equal, $AB = DC$ and $AC = DE$.

The area of the quadrilateral *ADBE* is equal to

$$AD \times \frac{AB + DE}{2}, \text{ or } \frac{(AC + AB)^2}{2}.$$

But the area of the quadrilateral also

$$= \frac{\overline{CB}^2}{2} + \frac{2(AB \times AC)}{2}.$$

$$\therefore \frac{\overline{CB}^2}{2} + AB \times AC = \frac{(AC + AB)^2}{2},$$

or $\overline{CB}^2 + 2(AB \times AC) = \overline{AC}^2 + 2(AB \times AC) + \overline{AB}^2.$

$$\overline{CB}^2 = \overline{AC}^2 + \overline{AB}^2.$$

This beautiful proof was worked out by General James A. Garfield. It was printed in the *New England Journal of Education* in or about the year 1876, several years before General Garfield became President.

20. PROOF NUMBER XI.

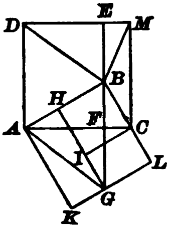


Fig. 20.

The parallelogram $ADEF$ = parallelogram $ADBG$.

Area of triangle $ABG = \frac{1}{2}$ of square AL, but triangle $ABG = \frac{1}{2}$ parallelogram $ADBG$.

$\therefore$ square AL = parallelogram $ADGB$ = parallelogram $ADEF$.

In like manner show square BI = parallelogram $FEMC$.

But $ADEF + FEMC$ = square on AC.

$$\therefore \overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2.$$

21. PROOF NUMBER XII.

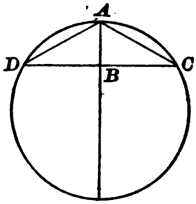


Fig. 21.

Let AB be the bisector of the angle DAC in the triangle DAC.

Then $AC \times AD = DB \times BC + \overline{AB}^2$.

But $AD = AC$;

$$DB = BC.$$

$$\therefore \overline{AC}^2 = \overline{BC}^2 + \overline{AB}^2.$$

22. PROOF NUMBER XIII.

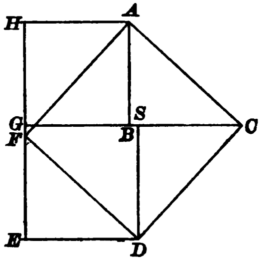


Fig. 22.

ABC is the right-angled triangle.

The square $AFDC$ = triangle ABC + triangle DSC + figure $FABSD$.

But triangle ABC = triangle AHF.

Triangle DSC = triangle FED.

Triangle AHF + triangle EFD + figure $FABSD$ $= \overline{AB}^2 + \overline{BC}^2$.

$$\therefore \overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2.$$

NOTE. — Monograph No. 2, which will appear shortly, will contain thirteen more proofs of this proposition, together with an interesting discussion of the attempts to square the circle.

JUST PUBLISHED

COMPLETE TRIGONOMETRY

PLANE AND SPHERICAL

BY WEBSTER WELLS, S.B.

Professor of Mathematics in the Massachusetts Institute of Technology

IN this new Trigonometry many improvements have been made, notably in the proofs of several of the functions, in the general demonstrations of the formulæ, in the solution of right triangles by natural functions, etc.

The book contains an unusually large number of examples. These have been selected with great care, and most of them are new. The results have been worked out by the aid of the author's New Four-Place Logarithmic Tables, which contain also Tables of Natural Functions.

Half leather. Pages, viii + 182. Price, 90 cents. With tables, $1.08.
New Plane Trigonometry, bound separately, 75 cents.

NEW PLANE AND SPHERICAL TRIGONOMETRY

BY WEBSTER WELLS, S.B.

Professor of Mathematics in the Massachusetts Institute of Technology

THIS is a more advanced work than the Complete Trigonometry, and is intended for the use of technical schools, colleges, and universities. The problems call for the use of Six-place Tables.

The systematic arrangement, clear statement of principles, rigorous and concise proofs and demonstrations, and large number of practical examples, combine to render the book one of the most satisfactory texts of its kind.

Half leather. Octavo. Pages, vi + 126. Price $1.00. With tables, $1.25.
With tables, and Robbins's Surveying and Navigation, $1.50.

D. C. HEATH & CO., Publishers, Boston, New York, Chicago

CHAPTER II.

THEOREM.

10. *The square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the other two sides.*

NOTE.—The history of this theorem, and thirteen of the proofs, are given in Heath's Mathematical Monograph, No. 1.

23. PROOF NUMBER XIV.

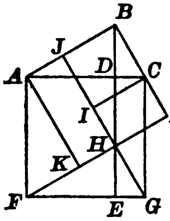


Fig. 23.

ABC is the right-angled triangle.

The square *AKLB* = parallelogram *AFHB*
= rectangle *AFED*.

The square *BJIC* = parallelogram *BHGC*
= rectangle *CGED*.

$\therefore AFGC = AKLB + BJIC.$

Or, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2.$

24. PROOF NUMBER XV.

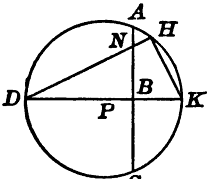


Fig. 24.

Let *AC* be any chord in the circle *DAC* with centre at *P*. Draw the diameter *DK* perpendicular to the chord AC, and join any point in the circumference, as H, with D and K.

We have $DN : DK :: DB : DH$.

$$\therefore DN \times DH = DK \times DB.$$

$$\text{But } DK = DB + BK.$$

$$\begin{aligned}\text{Hence } DN \times DH &= DB(DB + BK) \\ &= \overline{DB}^2 + DB \times BK.\end{aligned}$$

$$\text{But } DB \times BK = AB \times BC = \overline{AB}^2.$$

$$\therefore DN \times DH = \overline{DB}^2 + \overline{AB}^2.$$

Now conceive DH to revolve about D as a centre until the point H coincides with the point A; then will $DN = DH = DA$,

and $$\overline{DA}^2 = \overline{DB}^2 + \overline{AB}^2.$$

25. PROOF NUMBER XVI.

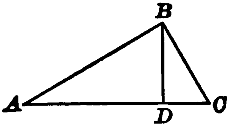


Fig. 25.

Make $AB = x$, $BC = y$, $CA = z$, $AD = m$, $CD = n$, and $DB = w$;

then $$x^2 : y^2 :: \tfrac{1}{2}(mw) : \tfrac{1}{2}(nw). \qquad (1)$$

Also $$z^2 : y^2 :: \tfrac{1}{2}(xy) : \tfrac{1}{2}(nw). \qquad (2)$$

Taking proportion (1) by composition,

$$x^2 + y^2 : y^2 :: \tfrac{1}{2}(m + n)w : \tfrac{1}{2}(nw) \qquad (3)$$

Since $\frac{1}{2}(xy) = \frac{1}{2}(m + n)w$, proportion (2) becomes

$$z^2 : y^2 :: \tfrac{1}{2}(m + n)w : \tfrac{1}{2}(nw) \qquad (4)$$

From proportions (3) and (4) by equal ratios,

$$x^2 + y^2 : y^2 :: z^2 : y^2;$$

or,

$$y^2z^2 = y^2(x^2 + y^2).$$

$$\therefore\ z^2 = x^2 + y^2.$$

26. PROOF NUMBER XVII

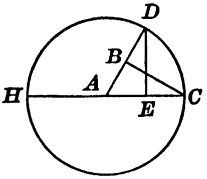


Fig. 26.

ABC is the right-angled triangle.

Using the hypotenuse *AC* as radius and with centre at *A* describe a circle. Prolong *AB* until it meets the circumference in *D*. From *D* let fall the perpendicular *DE* upon *AC*.

Triangle ADE = triangle ABC, $DE = CB = a$,

$$HA = CA = h,\ EA = BA = b,\ HE = h + b,$$

and

$$CE = h - b.$$

Since DE is a mean proportional betwen HE and CE, we have $\overline{DE}^2 = HE \times CE$.

$$\therefore\ a^2 = (h + b) \times (h - b) = h^2 - b^2,$$

or,
$$h^2 = a^2 + b^2.$$

27. PROOF NUMBER XVIII.

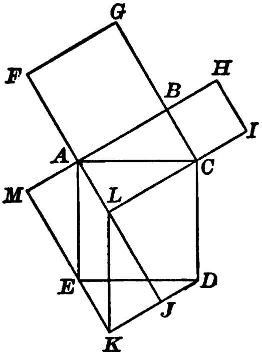


Fig. 27.

ABC is the right-angled triangle.

It is evident that square $ACDE$ = parallelogram $CDKL$ + parallelogram $AEKL$.

But parallelogram $CDKL$ = square $ABGF$, and parallelogram $AEKL$ = square $BCIH$.

$$\therefore\ \overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2.$$

28. PROOF NUMBER XIX.

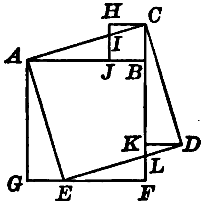


Fig. 28.

ABC is the right-angled triangle.

Triangle *AEG* = triangle *CDK*.

Triangle *ELF* = triangle *AIJ*.

Triangle *CHI* = triangle *DKL*.

$$\therefore \overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2.$$

29. PROOF NUMBER XX.

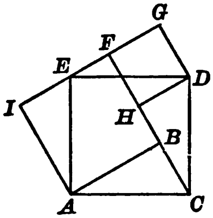


Fig. 29.

ABC is the right-angled triangle.

Let $AB = b$, $BC = a$, and $CA = h$;

then $AEFHDC = AEDC + DEG - DGFH$,

$$= h^2 + \tfrac{1}{2}ab - a^2. \qquad (1)$$

Also $AEFHDC = ABC + ABFI + CDH - AEI,$

$$= \tfrac{1}{2}ab + b^2 + \tfrac{1}{2}ab - \tfrac{1}{2}ab. \qquad (2)$$

Equating (1) and (2), and solving for h^2, we have

$$h^2 = a^2 + b^2.$$

30. PROOF NUMBER XXI.

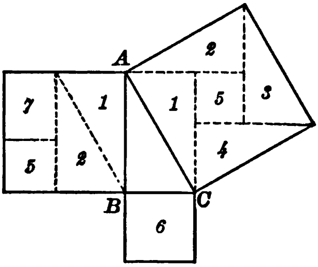


Fig. 30.

Since $1 = 1$, $2 = 2$, $5 = 5$, and $3 + 4 = 6 + 7$, it is evident that

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2.$$

31. PROOF NUMBER XXII.

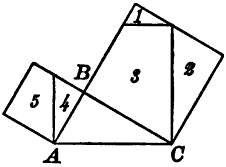


Fig. 31.

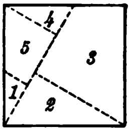


Fig. 32.

Obviously, $1 = 1$, $2 = 2$, $3 = 3$, $4 = 4$, and $5 = 5$; that is,

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2.$$

32. PROOF NUMBER XXIII.

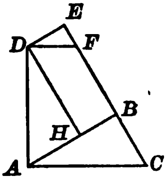


Fig. 33.

Let ABC be the right-angled triangle.

Put $AB = b,\ BC = a,\ AC = h.$

Erect AD perpendicular to and equal to AC.
Erect BE perpendicular to and equal to AB.
Connect D and E. Draw DF perpendicular to AD, and DH perpendicular to AB.

Now

$$\text{triangle } ADH = \text{triangle } ABC,$$

$$AH = BC = a, \text{ and } BH = DE = b - a.$$

Since the triangles ABC and DEF are similar, we have,

$$DF = h\left(1 - \frac{a}{b}\right), \text{ and } EF = a\left(1 - \frac{a}{b}\right).$$

Obviously,

$$ADEC = ADEB + ABC$$

$$= \tfrac{1}{2}b\left[b + (b - a)\right] + \tfrac{1}{2}ab = b^2. \quad (1)$$

Also, $ADEC = ADFC + DEF,$

$$= \frac{h}{2}\left[h\left(1 - \frac{a}{b}\right) + h\right] + \frac{ab}{2}\left(1 - \frac{a}{b}\right)^2. \qquad (2)$$

Equating (1) and (2), and solving for h^2,

$$(2b - c)h^2 = (2b - a)(a^2 + b^2).$$

$$\therefore\ h^2 = a^2 + b^2.$$

Edward Olney, in his *Elements of Geometry*, Part III., gives the following proof.

33. PROOF NUMBER XXIV.

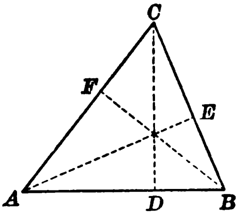


Fig. 34.

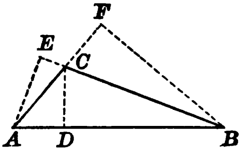


Fig. 35.

ABC is any triangle. Let *AE*, *BF*, and *CD* be the three perpendiculars from the angles upon the opposite sides, or upon the sides produced. It will at once be seen that a circumference described on any side as a diameter passes through the feet of two of the perpendiculars. Recalling to mind the theorems relating to secants and to intersecting chords, we readily obtain the following:

$$AB \times AD = AC \times AF = \overline{AC}^2 \pm AC \times FC,$$

and

$$AB \times DB = BC \times EB = \overline{BC}^2 \pm BC \times CE;$$

adding,

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2 \pm 2\,AC \times FC \text{ (or } 2\,BC \times CE),$$

the + sign being taken when C is obtuse, and the − sign when C is acute. If, however, C is a right angle, CE and CF become 0, whence

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2.$$

The two proofs given below were originated — the first in 1875, the second in 1877 — by George M. Philips, Ph.D., Principal of the West Chester State Normal School. Through the courtesy of Dr. Philips, these demonstrations are inserted here.

34. PROOF NUMBER XXV.

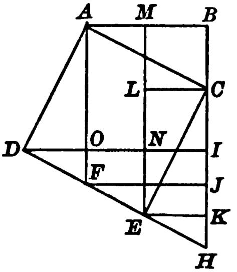


Fig. 36.

ABC is the right-angled triangle. Construct square $ACED$. Draw perpendiculars AF, EM, CL, DI, FJ, EK, having prolonged BC and DE until they met.

$$AOIB = \text{square on } AB, \quad \because \text{ triangle } ABC$$
$$= \text{triangle } AOD.$$
$$NIKE = MBCL = \text{square on } BC,$$
$$\because NE = EK \, (DNE = EKC);$$
$$\therefore NIKE \text{ is a square, and}$$
$$NI = LC \text{ and } KI = CB, \quad \because BI = CK \; (= AO).$$
$$\therefore \text{(axiom) } KI = CB. \quad AFHC = ADEC.$$
$$DNE = FJH. \quad ABC = FJH.$$
$$DOF = EKH.$$
$$NEKI = OFJI, \quad \because DIH - DNE - EHK$$
$$= DIH - FJH - DOF.$$
$$\therefore AFHC \; (= \text{square } ACED)$$
$$= \overbrace{AOIC + FJH}^{\text{square } AB} + \overbrace{OIJF}^{\text{square } BC}$$

The reader will, we think, find no proof in this entire chapter that in brevity and beauty surpasses the following, originated by Dr. Philips in 1877.

35. Proof Number XXVI.

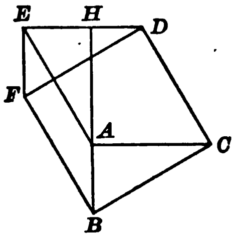


Fig. 37.

ABC is the right-angled triangle. Construct a square, *BCDF*, on the hypotenuse. From *A* draw *AE* parallel and equal to *BF* (and *CD*). Draw *ED*, *EF*, and prolong *BA* to *H*.

EABF and *EACD* are parallelograms.

$$FED = BAC, \quad \therefore\ BFEDCA = \text{square } BFDC.$$

$$\text{Triangle } AHE = \text{triangle } FDE.$$

Since altitude of *BAEF*

$$= HE = AB, \quad BAEF = \overline{AB}^2.$$

Since altitude of $ACDE = AH = AC$.

$$ACDE = \overline{AC}^2,$$

$$\therefore\ BFEDCA = \overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2.$$

CHAPTER III.

THE VALUE OF π.

36. THE arithmetical value of π, the ratio of the circumference of a circle to its diameter, is 3.141592653589793238462, and so on in an endless fraction. But just as there are persons who will persist in believing that the earth is flat, so there are persons who believe that it is possible to obtain the exact value of π. In other words, they believe that a square can be constructed which shall be exactly equivalent in area to a given circle. They hold this belief, too, in spite of the fact that Johann Heinrich Lambert, as early as 1750, proved π incommensurable, *i.e.* inexpressible in a finite fraction. Since 1882, when Lindemann proved that π is also transcendental, *i.e.* cannot be represented as the root of any algebraic equation with rational coefficients, the circle-squarers have, if possible, a more hopelessly hard wall against which to dash their heads.

The ancient geometricians were, of course, ignorant of these facts, for it was left for modern algebra to show why the circle cannot be squared. Antiphon (circ. 420 B.C.) and Hippocrates (circ. 470 B.C.) may therefore be excused if, in the dim mathematical light of their day, they attempted the impossible; but there is no excuse for the cyclometers who, in the light of the present, bring themselves out on parade. The mathematical acquirements of nineteenth century circle-squarers, it is true, do not often extend far beyond long division; but for knowledge of the subject that has engaged their attention they have substituted dogged persistency in their chosen work, and virulence in attack upon those who, in the kindness of their hearts, have striven to free them from the snares into which they have fallen.

"The feeling which tempts persons to this problem," says the distinguished mathematician De Morgan, "is that which, in romance, made it impossible for a knight to pass a castle which belonged to a giant or an enchanter. This rinderpest of geometry cannot be cured when once it has seated itself in the system. All that can be done is to apply what the learned call prophylactics to those who are yet sound. When once the virus gets into the brain, the victim goes round the flame like a moth, — first one way and then the other, beginning again where he ended, and ending where he began."

37. Eudemus, a distinguished pupil of Aristotle, speaks, in his *History of Geometry*, in the following manner of two early attempts to square the circle: "Amongst the many persons who have sought the squaring of the circle, both Antiphon and Hippocrates believed that they had found it, and were equally mistaken."

Before giving the method by which these distinguished men endeavored to solve this famous problem, we must pause long enough to speak briefly of their character and work. Antiphon and Hippocrates were, for their day and generation, geometricians worthy of the highest honors; and they are not for one moment to be classed with common modern circle-squarers.

Antiphon did himself great credit by showing how to construct a succession of regular polygons of 8, 16, 32, and 64 sides, and so on, of which each approaches nearer to the circle than the previous one. This part of his work is sound, and is found in the text-books on geometry to-day.

Hippocrates of Chios was one of the ablest of Greek geometricians. It was he who wrote the first elementary text-book on geometry, — a book on which Euclid's Elements was probably founded. It was Hippocrates also who denoted the square on a line by the word δύναμις, and thus gave the technical meaning to the word *power*, which it still retains in algebra. There is, indeed, reason for believing that the proposition known as Euc. I. 47 was at one time stated in the following form: "The total power of the sides of a right-angled triangle is the same as that of the hypotenuse." "Hippocrates discovered," says Professor Ball, "that similar segments of a circle contain equal angles; that the angle subtended by the chord of a circle is greater than, equal to, or less than a right angle as the segment of the circle containing it is less than, equal to, or greater than a semicircle; and probably several other of the propositions in the third book of Euclid."

38. The following account of the manner in which Antiphon sought to square the circle is given by Eudemus: —

"Having drawn a circle, Antiphon inscribed in it one of those polygons which can be inscribed, — let it be a square. Then he bisected each side of the square, and through the points of section drew straight lines at right angles to them, producing them to meet the circumference. These lines evidently bisect the corresponding segments of the circle. He then joined the new points of section to the ends of the sides of the square, so that four triangles were formed, and the whole inscribed figure became an octagon. And again, in the same way, he bisected each of the sides of the octagon, and drew from the points of bisection perpendiculars; he then joined the points where these perpendiculars met the circumference with the extremities of the octagon, and thus formed an inscribed figure of sixteen sides. Again, in the same manner, bisecting the sides of the inscribed polygon of sixteen sides, and drawing straight lines, he formed a polygon of twice as many sides; and doing the same again and again, until he had exhausted the surface, he concluded that in this manner a polygon would be inscribed in the circle, the sides of which, on account of their minuteness, would coincide with the circumference of the circle. But we can substitute for each polygon a square of equal surface; therefore we can, since the surface coincides with the circle, construct a square equal to a circle."

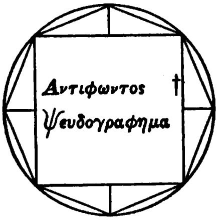


Fig. 38.

† Antiphon's false delineation.

Simplicius, who quotes the above from Eudemus's

History of Geometry, offers the following comment upon Antiphon's reasoning: "It is a principle that a straight line cannot coincide with a circumference, for one without meets the circle in one point only, one within in two points, and not more, and the meeting takes place in single points. Yet, by continually bisecting the space between the chord and the arc, it will never be exhausted, nor shall we ever reach the circumference of the circle, even though the cutting should be continued *ad infinitum.*"

39. PARALOGISM OF HIPPOCRATES.

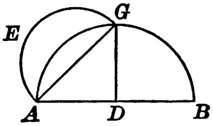


Fig. 39.

Let a semicircle ABG be described on the straight line AB; bisect AB in D; from the point D draw a perpendicular DG to AB, and join AG; this will be the side of the square inscribed in the circle of which ABG is the semicircle. On AG describe the semicircle AEG. Now, since the square on AB is equal to double the square on AG (and since the squares on the diameters are to each other as the respective circles or semicircles), the semicircle *AGB* is double the semicircle *AEG*. The quadrant *AGD* is therefore equal to the semicircle *AEG*. Take away the common segment lying between the circumference *AG* and the side of the square; then the remaining lune *AEG* will be equal to a square. Having thus shown that the lune can be squared, Hippocrates next tries, by means of the preceding demonstration, to square the circle thus: —

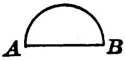


Fig. 40.

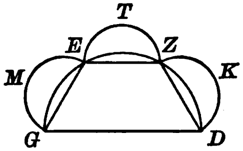


Fig. 41.

Let there be a straight line *AB*, and let a semicircle be described on it; take *GD* double of *AB*, and on it also describe a semicircle; and let the sides of a hexagon, *GE*, *EZ*, and *ZD*, be inscribed in it. On these sides describe the semicircles *GME*, *ETZ*, *ZKD*. Then each of these semicircles described on the sides of the hexagon is equal to the semicircle on *AB*, for *AB* is equal to each side of the hexagon. The four semicircles are equal to each other, and together are then four times the semicircle on *AB*; but the semicircle on *GD* is also four times that on *AB*. The semicircle on *GD* is therefore equal to the four semicircles on the sides of the hexagon. Take away from the semicircles on the sides of the hexagon and from that on *GD* the common segments contained by the sides of the hexagon and the periphery of the semicircle *GD*; the remaining lunes, *GME, ETZ*, and *ZKD*, together with the semicircle on *AB*, will be equal to the trapezium *GE, EZ, ZD*. If we now take away from the trapezium the excess, that is, a surface equal to the lunes (for it has been shown that there exists a rectilineal figure equal to a lune), we shall obtain a remainder equal to the semicircle *AB*; we double this rectilineal figure which remains, and construct a square equal to it. That square will be equal to the circle of which *AB* is the diameter, and thus the circle has been squared.

Upon this effort to square the circle Eudemus makes the following observation: "The treatment of the problem is indeed ingenious; but the wrong conclusion arises from assuming that as demonstrated generally which is not so; for not every lune has been shown to be squared, but only that which stands over the side of the square inscribed in the circle; but the lunes in question stand over the sides of the inscribed hexagon."

If generalizations too broad for the foundations upon which they rest were confined to circle-squarers, we should have great reason to rejoice.

40. Aristotle and his commentators mention another Grecian mathematician, Bryson, who attempted to square the circle. Bryson inscribed a square or, more generally, any regular polygon, in a circle, and circumscribed another of the same number of sides about the circle; he then argued that the circle is larger than the inscribed and less than the circumscribed polygon. Thus far Bryson was on firm ground, but at this point he made a mistake. He erroneously assumed that the excess of area in the case of the circumscribed polygon is equal to the deficiency in the case of the inscribed polygon, and then concluded that the circle is the mean between the two.

"It seems, too," says Allman, in his *History of Greek Geometry*, "that some persons who had no knowledge of geometry took up the question and fancied that they should find the square of the circle in surface measure if they could find a square number which is also a cyclical number, — numbers as 5 or 6, whose square ends with the same number, are called by arithmeticians cyclical numbers." Hankel, the German mathematician, speaking of this scheme for squaring the circle, says it is unfortunate that we cannot regard it as a joke. Those who proposed this solution were in earnest. Because of false solutions by persons who merely pretended to be geometricians, an element of ridicule was attached to circle squaring even in these early days. Aristophanes, in his comedy of *The Birds*, makes fun of the idea.

"Although the attempts of Antiphon and Bryson to square the circle," says Allman, "did not meet with much favor from the ancient geometers, and were condemned on account of the paralogisms in them, yet their conceptions contain the first germ of the infinitesimal method. To Antiphon is due the merit of having first got into the right track by introducing for the solution of this problem — in accordance with the atomic theory then nascent — the fundamental idea of infinitesimals, and by trying to exhaust the circle by means of inscribed polygons of continually increasing number of sides. Bryson is entitled to praise for having seen the necessity of taking into consideration the circumscribed as well as the inscribed polygon, and thereby obtaining a superior as well as an inferior limit to the area of the circle. Bryson's idea is just, and should be regarded as complementary to the idea of Antiphon, which it limits and renders precise. Later, after the method of exhaustions had been invented, in order to supply demonstrations which were perfectly rigorous, the two limits, inferior and superior, were always considered together, as we see in Euclid and Archimedes."

41. In the beginning of this chapter we spoke of the ignorance and virulence shown by modern circle-squarers. We are now ready to give a few choice specimens of these qualities.

A certain gentleman took it into his head that the Lord Chancellor of England had offered £100,000 reward to any one who should square the circle. Quickly and neatly the gentleman squared it, and then, very properly, hied him to London to claim the trifling reward. Hearing that De Morgan was a great mathematician, he left his papers with him, together with a polite note to the Chancellor asking him kindly to send him a check for the £100,000. De Morgan returned the papers with a note saying that no such reward had been offered. This was bad, but what followed was worse. Continuing, De Morgan pointed out, in as gentle a manner as possible, that the gentleman had not sufficient knowledge to see in what the problem consisted. This brought from the circle-squarer the following letter:—

DOCTOR MORGAN:

Sir,—Permit me to address you.

Brute Creation may perhaps enjoy the faculty of beholding visible things with a more penetrating eye than ourselves. But Spiritual objects are as far out of their reach as though they had no being.

Nearest therefore to the brute Creation are those men who Suppose themselves to be so far governed by external objects as to believe nothing but what they See and feel And Can accomedate to their Shallow understanding and imaginations

My Dear Sir Let us all consult ourselves by the wise proverb. I believe that every man^{s} merit & ability aught to be appreciated and valued In proportion to its worth & utility.

In whatever State or Circumstances they may fortunately or unfortunately be placed

And happy it is for every man to know his worth and place

When a gentleman of your Standing in Society Clad with those honors Can not understand or Solve a problem That is explicitly explained by words and Letters and mathematically operated by figures He had best consult the wise proverb

Do that which thou Canst understand and Comprehend for thy good. I would recommend that Such Gentleman Changed his business

And appropriate his time and attention to a Sunday School to Learn what he could and keep the little Children from durting their Close.

With Sincere feeling of Gratitude for your weakness and Inability I am

Sir your Superior in Mathematics——

1849 June th 29.

A few days after the reception of this elegant epistle De Morgan was favored with another from the same source. We give it below: —

DR MORGIN SIR

I wrote and Sent my work to Professor ——— of ——— State of ——— United States

I am now in the possession of the facts that he highly approves of my work. And Says he will Insure me Reward in the States

I write this that you may understand that I have knowledge of the unfair way that I am treated In my own nati County

I am told and have reasons to believe that it is the Clergy that treat me so unjust.

I am not Desirous of heaping Dishonors upon my own Nation. But if I have to Leave this Kingdom without my Just dues, The world Shall know how I am and have been treated.

I am Sir Desirous of my

Just dues ——

1849 JULY 3.

But De Morgan's self-control was to have another test. The circle-squarer's friends hastened to his support. One of them delivered himself in the following manner: —

July 7th, 1849.

Sir, I have been given to understand that a friend of mine one whom I shall never be ashamed to acknowledge as such tho' lowly his origine; nay not only not ashamed but proud of doing so for I am one of those who esteem and respect a man accoiding to his ability and probity, deeming with Dr. Watts that the mind is the standard of the man, has laid before you and asked your opinion of his extraordinary performance, viz. the quadrature of the circle, he did this with the firmest belief that you would not only treat the matter in a straightforward manner but with the conviction that from your known or supposed knowledge of mathematicks would have given an upright and honorable decision upon the subject; but the question is have you done so? Could I say yes I would with the greatest of pleasure and have congratulated you upon your decision whatever it might have been but I am sorry to say that I cannot your letter is a paltry evasion, you say that it is a great pity that you (Mr. ———) should have attempted this (the quadrature of the circle) for your mathematical knowledge is not sufficient to make you know in what the problem consists, you don't say in what it does consist according to your ideas, oh! no nothing of the sort, you enter into no disquisition upon the subject in order to show where you think Mr. ——— is wrong and why you have not is simply — because you cannot — you know that he has done it and what is if I am not wrongly informed you have been heard to say so. He has done what you nor any other mathematician as those who call themselves such have done. And what is the reason that you will not candidly acknowledge to him as you have to others that he has squared the circle shall I tell you? it is because he has performed the feat to obtain the glory of which mathematicians have battled from time immemorial that they might encircle their brows with a wreath of laurels far more glorious than ever conqueror won it is simply this that it is a poor man a humble artisan who has gained that victory that you don't like to acknowledge it you don't like to be beaten and worse to acknowledge that you have miscalculated, you have in short too small a soul to acknowledge that he is right.

I was asked my opinion and I gave it unhesitatingly in the affirmative and I am backed in my opinion not only by Mr. ——— a mathematician and watchmaker residing in the boro of Southwark but by no less an authority than the professor of mathematics of ——— College ——— ——— United States Mr. ——— and I presume that he at least is your equal as an authority and Mr. ——— says that the government of the U. S.

will recompense M. D. for the discovery he has made if so what a reflection upon Old england the boasted land of freedom the nursery of the arts and sciences that her sons are obliged to go to a foreign country to obtain that recompense to which they are justly entitled.

In conclusion I had to contradict an assertion you made to the effect that there is not nor ever was any reward offered by the government of this country for the discovery of the quadrature of the circle. I beg to inform you that there was but that it having been deemed an impossibility the government has withdrawn it. I do this upon no less an authority than the Marquis of Northampton.

I am, Sir, yours

DR. MORGAN

While circle-squarers have rarely agreed among themselves concerning the numerical value of π, a few have fought long and valiantly for $3\frac{1}{8}$. Mr. James Smith, of Liverpool, member of the Mersey Docks and Harbor Board, attributes the first announcement of $3\frac{1}{8}$ as the ratio of the circumference of a circle to its diameter to M. Joseph Lacomme, a French well-sinker. "It does the well-sinker," says De Morgan, "great honor, being so near the truth, and he having no means of instruction. In our day, when an ignorant person chooses to bring his fancy forward in opposition to demonstration which he will not study, he is deservedly laughed at."

Speaking of this same James Smith, Dr. Morgan says, "He is, beyond a doubt, the ablest head at unreasoning, and the greatest hand at writing it, of all who have tried in our day to attach their names to an error. Common cyclometers sink into puny orthodoxy by his side. The behavior of this singular character induces me to pay him the compliment Achilles paid Hector, — to drag him around the walls again and again."

We have not space in which to have him taken around the walls many times, but we will make room enough to allow Dr. De Morgan to drag him around once as follows: —

"As to Mr. James Smith, we can only say this: he is not mad. Madmen reason rightly upon wrong premises; Mr. Smith reasons wrongly upon no premises at all. His procedures are not caricature of reasoning; they are caricature of blundering. The old way of proving that $2 = 1$ is solemn earnest compared with his demonstration. As follows: —

$$\text{Let } X = 1.$$
$$\text{Then } X^2 = X.$$
$$\text{And } X^2 - 1 = X - 1.$$

Divide both sides by $X - 1$; then

$$X + 1 = 1;\ \text{but } X = 1, \text{ whence } 2 = 1."$$

The reader perhaps wonders why a great mathematician like De Morgan would waste precious time upon circle-squarers. Here is the answer:—

"I now reply to a question I have been asked again and again: 'Why do you take so much trouble to expose such a reasoner as Mr. Smith?' I answer, as a deceased friend of mine used to answer on like occasions, a man's capacity is no measure of his power to do mischief. Mr. Smith has untiring energy, which does something; self-evident honesty of conviction, which does more; and a long purse, which does most of all. He has made at least ten publications, full of figures which few readers can criticise. A great many people are staggered to this extent, that they imagine there must be the indefinite *something* in the mysterious *all this*. They are brought to the point of suspicion that the mathematicians ought not to treat all this work with such undisguised contempt. Now, I have no fear for π; but I do think it possible that general opinion might in time demand that the crowd of circle-squarers, etc., should be admitted to the honors of opposition; and this would be a time-tax of five per cent, one man with another, upon those who are better employed. Mr. James Smith may be made useful, in hands which understand how to do it, toward preventing such opinion from growing. A speculator who expressly assumes what he wants to prove, and argues that all which contradicts it is absurd *because* it cannot stand side by side with his assumption, is a case which can be exposed to all."

De Morgan concludes by saying it will probably be thought that in all his thrusts at poor Mr. Smith there may be a little of the spirit of Colonel Quagg, who said, "I licks ye because I kin, and because I like, and because ye'se critters that licks is good for."

De Morgan having failed most signally in his efforts to convince Mr. Smith of his error, William Whewell, the distinguished philosopher and scholar, decided to see what he could do. He therefore wrote the following letter to the great circle-squarer: —

THE LODGE, CAMBRIDGE, Sept. 14th, 1862.

Sir, — I have received your explanation of your proposition that the circumference of the circle is to its diameter as 25 to 8. I am afraid I shall disappoint you by saying that I see no force in your proof; and I should hope that you will see that there is no force in it if you consider this: In the whole course of the proof, though the word circle occurs, there is no property of the circle employed. You may do this: you may put the word hexagon or dodecagon, or any other word describing a polygon, in the place of circle in your proof, and the proof would be just as good as before. Does not this satisfy you that you could not have proved a property of that special figure, — a circle?

Or you may do this: calculate the side of a regular polygon of 24 sides inscribed in a circle. I think you are mathematician enough to do this. You will find that if the radius of the circle be 1, the side of this polygon is .264, etc. Now, the arc which this side subtends is, according to your proposition, $\frac{3.125}{12} = .2604$, and therefore the chord is greater than its arc, which you will allow is impossible.

I shall be glad if these arguments satisfy you, and I am, sir,

Your obedient servant,

W. Whewell.

Although Whewell's proof that the inveterate circle-squarer's ratio was wrong is so simple that a schoolboy can work it out with ease, Mr. Smith, so far from acknowledging his error, made a violent attack upon Whewell, claiming that he had put "the Master of Trinity in the stocks without hope of escape," and clung to his idol — $3\frac{1}{8}$ — more tenaciously than before.

FAMOUS
GEOMETRICAL THEOREMS
AND PROBLEMS

WITH THEIR HISTORY

BY

WILLIAM W. RUPERT, C.E.
SUPERINTENDENT OF SCHOOLS, POTTSTOWN, PA.

> "I am sure that no subject loses more than mathematics by any attempt to dissociate it from its history." — J. W. L. GLAISHER.

BOSTON, U.S.A.
D. C. HEATH & CO., PUBLISHERS
1901

Copyright, 1901,
By D. C. Heath & Co.

CHAPTER IV.

TRISECTION OF AN ANGLE.

42. One of the most celebrated geometrical problems of antiquity was the trisection of an angle. It stands side by side with those other famous problems, — the squaring of the circle and the duplication of the cube.

The bisection of an angle is one of the easiest problems in geometry. Any boy who has studied geometry a few months is familiar with the problem and with its solution. The trisection of an angle, however, presented great and unexpected difficulties. The Pythagoreans readily divided a right angle into three equal parts, but the general problem, though easy in appearance, baffled the powers of the ablest geometricians of that day. No wonder the early Greek mathematicians found the solution of the problem beyond their reach, for modern analysis shows that the trisection of an angle is an insoluble problem if in our constructions we confine ourselves to the use of circles and straight lines, *i.e.* to Euclidean geometry.

Let us now consider why the trisection of an angle transcends elementary geometry.

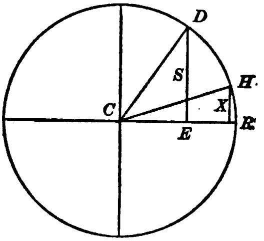


Fig. 42.

Let DCR be any angle, and let the angle HCR be one-third of it. Represent the angle HCR by a. Let S be the sine of the given angle, and let X be the sine of the angle a.

Now from trigonometry we have,

$$\sin(a+b) = \sin a \cos b + \cos a \sin b,$$

and $\cos(a+b) = \cos a \cos b - \sin a \sin b$.

Making $b = a$,

$$\sin 2a = 2 \sin a \cos a,$$

and $\cos 2a = \cos^2 a - \sin^2 a$.

Making $b = 2a$,

$$\sin 3a = \sin a (\cos 2a) + \cos a (\sin 2a).$$

$$\therefore \sin 3a = \sin a(\cos^2 a - \sin^2 a) + \cos a(2 \sin a \cos a)$$

$$= \sin a \cos^2 a - \sin^3 a + 2 \sin a \cos^2 a$$

$$= \sin a (\cos^2 a - \sin^2 a + 2 \cos^2 a)$$

$$= \sin a(3 \cos^2 a - \sin^2 a).$$

By substitution we now have

$$S = X(3 \cos^2 a - \sin^2 a);$$

but $\cos^2 a = 1 - \sin^2 a$,

$$\therefore S = X(3 - 3 \sin^2 a - \sin^2 a) = X(3 - 4X^2),$$

or $4X^3 = 3X - S$.

Thus it is plain that the trisection of an angle, when considered analytically, requires the solution of a cubic equation; and since a construction by means of circles (whose equations are of the form $x^2 + y^2 + ax + by + c = 0$) and straight lines (whose equations are of the form $\alpha x + \beta y + \gamma = 0$) cannot be equivalent to the solution of a cubic equation, we reach the conclusion that the trisection of an angle is beyond the reach of elementary geometry. By the use of the conic sections, however, the problem is readily solved in many ways.

43. Pappus, a Grecian mathematician who lived and taught at Alexandria about the end of the third century, has given us the following beautiful reduction of the problem which we are now discussing. Professor Ball in his *History of Mathematics* says that the reduction is quoted by Pappus, but adds that he does not know to whom it was due originally.

"Since we can trisect a right angle," says Pappus, "it follows that the trisection of any angle can be effected if we can trisect an acute angle."

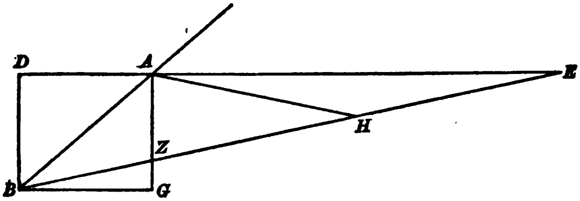


Fig. 43.

Let *ABG* be the given acute angle which it is required to trisect.

From any point *A* on the line *AB*, which forms one leg of the given angle, let fall a perpendicular *AG* on the other leg, and complete the rectangle *AGBD*.

Suppose now that the problem is solved, and that a line is drawn making with *BG* an angle which is the third part of the given angle *ABG*; let this line cut *AG* in *Z*, and be produced until it meet *DA* produced in the point *E*. Let the straight line *ZE* be bisected in *H*, and *A* joined with *H*; then the lines *ZH*, *HE*, *AH*, and *BA* are all evidently equal to each other, and, therefore, the line *ZE* is double the line *AB*, which is known.

The problem of the trisection of an angle is thus reduced to another: —

From any vertex of a rectangle *BDAG* draw a line *BE*, so that the part *ZE* of it intercepted between the two opposite sides, one of which is produced, shall be equal to a given line.

But the problem in its reduced form was just as far beyond the reach of Euclidean geometry as was the problem in its original form. The mathematicians went round and round the problem; they advanced to the attack from a thousand directions, but after all they found themselves moving in the circumference of a circle and were as far as ever from the centre, — the coveted solution.

44. While the geometricians quoted by Pappus could not solve the problem, Pappus himself, who lived at a time when the conic sections had been developed to some extent, fixed the position of the point *E* by means of an hyperbola. Pappus also claims, as his own, a solution by means of the conchoid of Nicomedes.

We may here premise that the conchoid is the locus of a point in a line which revolves on and slides in a fixed pivot, so as to allow a constant portion of the line to project beyond a fixed right line. In the following figure, B is the point in the line, C is the fixed pivot, XY is the fixed right line, AB is the constant portion projecting beyond the right line, and $DB''BB'K$ is the conchoid. This word signifies shell-like, and it will be observed that the form of the curve justifies its name.

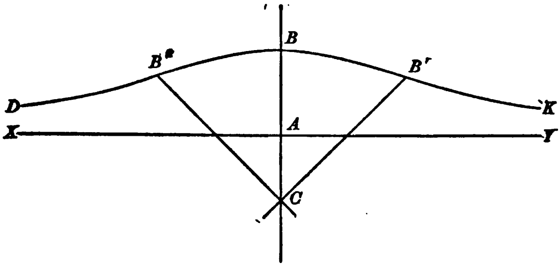


Fig. 44.

By the aid of the conchoid we can fix the position of the point E, and thus trisect the angle ABG, in the following manner:

From B as pole, with AG as fixed right line, and $2\,AB$ as constant distance, describe a conchoid which shall meet DA produced in E.

The line BE cuts AG in Z.

Bisect ZE in H, and join AH.

It is evident that $AH = HE = AB$, and the triangles AHE, ABH are isosceles.

Therefore the exterior angle $AHB = 2\,AEH = 2\,HBG$, and the angle $AHB = ABH = 2\,HBG$.

45. "Another method of trisecting an angle," says Allman in his *History of Greek Mathematics*, "is preserved in the works of Archimedes, being indicated in Prop. 8 of the *Lemmata* — a book which is a translation into Latin from the Arabic."

Prop. 8 of the *Lemmata* runs as follows:

"If a chord AB of a circle be produced until the part produced, BC, is equal to the radius; if then the point C be joined to the centre of the circle, which is the point D, and if CD, which cuts the circle in F, be produced until it cut it again in E, the arc AE will be three times the arc BF."

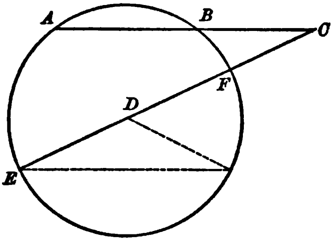


Fig. 45.

This theorem, the demonstration of which is perfectly plain, suggests the following reduction of the problem:

46. With the vertex A of the given angle BAC a centre, and any line AC or AB as radius, let a circle be described. Suppose now that the problem is solved, and that the angle EAC is the third part of the angle BAC; through B let a straight line be drawn parallel to AE, and let it cut the circle again in G and the radius CA produced in F.

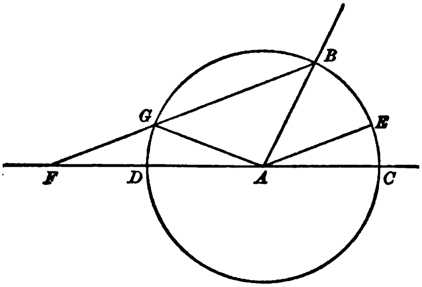


Fig. 46.

Then on account of the parallel lines AE and FGB, the angle ABG or the angle BGA, which is equal to it, will be double of the angle GFA. But the angle BGA is equal to the sum of the angles GFA and GAF. The angles GFA and GAF are, therefore, equal to each other, and consequently the lines GF and GA are also equal.

The problem is, therefore, reduced to the following:

From *B* draw the straight line *BGF*, so that the part of it, *GF*, intercepted between the circle and the diameter *CAD* produced shall be equal to the radius.

Here again, however, Euclidean geometry is incompetent to solve the problem, — it cannot fix the position of the point *F*. *F*, however, may be located either by the aid of the conchoid or by the use of the conic sections.

47. In conclusion we will give a solution of the trisection problem which was worked out by Hippias of Elis who lived about 420 B.C. This distinguished geometrician invented a transcendental curve by means of which he could divide an angle not only into three, but into any number of equal parts. This curve was called the quadratrix because it was also employed in an effort, though a vain one, to square the circle.

If the radius of a circle rotate uniformly round the centre *C* from the position *CE* through a right angle to *CA*, and in the same time a straight line drawn perpendicular to *CA* move uniformly parallel to itself from the position *CE* to *AF*, the locus of their intersection will be the quadratrix.

Let *CR* and *HB* be the position of these lines at any time, and let them cut in *P*, a point on the curve.

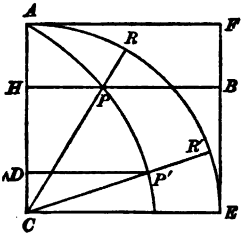


Fig. 47.

Then from the law controlling the moving lines we have

$$\text{angle } ECP : \text{angle } ECA :: CH : CA.$$

Similarly, if CR' be another position of the radius,

$$\text{angle } ECP' : \text{angle } ECA :: CD : CA.$$

$$\therefore \text{ angle } ECP : \text{angle } ECP' :: CH : CD;$$

$$\therefore \text{ angle } ECP' : \text{angle } P'CP :: CD : DH.$$

Hence, if the angle ECP be given, and it be required to divide it in any given ratio, say one to two, it is sufficient to divide CH in that ratio at D, and draw the line DP'; then CP' will divide ECP in the required ratio.

The quadratrix may easily be constructed from its polar equation. If CE be taken as the initial line, $CP = r$, the angle $ECP = \theta$, and $CE = a$, we have $\theta : \frac{1}{2}\pi :: r \sin\theta : a$, and the equation of the curve is $\pi r = 2a\theta \operatorname{cosec}\theta$.

Hippius devised an instrument with which he could construct the quadratrix mechanically, but Plato objected sternly to all constructions which involved the use of any mathematical instruments except a ruler and a pair of compasses. In this position, too, Plato was supported both by the best geometricians of his own day, and by those of a subsequent date. "The good of geometry," said Plato, speaking of mechanical solutions, "is set aside and destroyed, for we again reduce it to the world of sense, instead of elevating and imbuing it with the eternal and incorporeal images of thought, even as it is employed by God, for which reason He always is God."

CHAPTER V.

THE AREA OF A TRIANGLE IN TERMS OF ITS SIDES.

48. RULE. — *From half the sum of the three sides subtract each side separately; multiply together the half sum and the three remainders and extract the square root of the product.*

Modern text-books give, in substance, the following proof of the above rule.

PROOF NUMBER I.

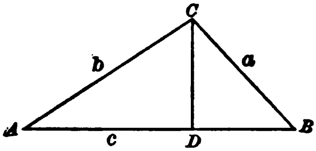


Fig 48.

Let a, b, c denote the sides of the triangle ABC. Then, by elementary geometry, we have

$$\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 - 2AB \times AD,$$

or $$a^2 = b^2 + c^2 - 2c \times AD;$$

whence $$AD = \frac{b^2 + c^2 - a^2}{2c}.$$

But $$\overline{CD}^2 = \overline{AC}^2 - \overline{AD}^2,$$

hence

$$\overline{CD}^2 = b^2 - \frac{(b^2 + c^2 - a^2)^2}{4c^2} = \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4c^2}.$$

Or $$CD = \frac{\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}}{2c}.$$

But the area

$$\frac{AB \times CD}{2} = \tfrac{1}{4}\sqrt{4b^2c^2 - (b^2 + c^2 - a^2)^2}.$$

Factoring the quantity under the radical sign (the difference of two squares), and then, in like manner, factoring again, we have the area

$$= \tfrac{1}{4}\sqrt{(b + c + a)(b + c - a)(a + b - c)(a - b + c)}.$$

Hence, if we put $S = \frac{a + b + c}{2}$, we shall have the

area $= \sqrt{S(S - a)(S - b)(S - c)}$.

We have been unable to determine who first discovered this proof. It seems to be general property now; and an older and very remarkable proof, which we give below, has evidently been forgotten or neglected.

PROOF NUMBER II.

49. The following proof of the rule under discussion in this chapter is remarkably ingenious and beautiful. Before entering upon the proof, however, we wish to say a word respecting its author, Hero of Alexandria.

This mathematician lived about 125 B.C. He was interested in science because of its practical applications; and he is credited with the construction of a powerful catapult, a stationary steam-engine, a double forcing pump, and many mechanical toys. That Hero was a mathematician of much more than ordinary ability may be gathered from the fact that he invented a solution (a mechanical one) of the Delian problem. Additional evidence of his power will be found in the demonstration which follows.

Let ABG be the given triangle.

Inscribe in it the circle DEH, having its centre at C. Draw the straight lines CA, CB, CG, CD, CE, and CH.

The rectangle $BG \times CE$ is double the triangle BCG; $AB \times CD$ is double ABC; $AG \times HC$ is double AGC.

Therefore the rectangle under CE and the perimeter of ABG is double the area of ABG.

Produce GB to T. Make $BT = AD$.

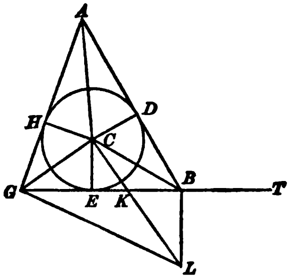


Fig. 49.

Then TG is half the perimeter.

Therefore the rectangle $TG \times CE$ is equal to the area of the triangle ABG.

Draw CL at right-angles to GC and BL to BG and join LG.

Then, the angles GCL, GBL, being each a right angle, the quadrilateral $GCBL$ is in a circle.

Therefore the angles GCB, GLB, are equal to two right angles, and also equal to the angles GCB, ACD, which also equal two right angles (since the angles at C are bisected by AC, BC, GC).

Therefore the angle ACD equals the angle GLB, and the triangles ACD, GLB are similar.

Therefore, $BG : BL :: AD : CD :: TB : CE$.

From this we have

$$BG : TB :: BL : CE :: BK : EK.$$

By composition, $GT : TB :: BE : EK$.

Hence,

$\overline{GT}^2 : GT \times TB :: BE \times EG : EG \times EK$ (or $\overline{CE}^2$).

Therefore, $\overline{GT}^2 \times \overline{CE}^2 = GT \times TB \times BE \times EG$.

But $GT \times CE$, which is equal to the area of the triangle, is the square root of $\overline{GT}^2 \times \overline{CE}^2$.

Therefore the area of the triangle is the square root of $GT \times TB \times BE \times EG$.

But each of these factors is given, for GT is half the perimeter; TB is half the perimeter minus BG, BE the same half minus AG, EG the same half minus AB.

PROOF NUMBER III.

50. Edward Olney, who was one of the few great teachers of mathematics, gives in his *Elements of Geometry*, in addition to the usual proof, the following beautiful demonstration.

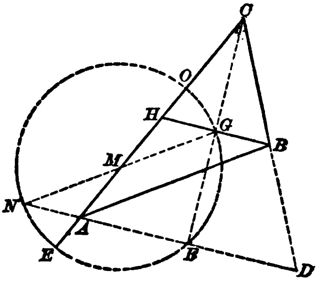


Fig. 50.

ABC is the given triangle.

Take $CH = CB$, $CD = CA$, and through M, the centre of HA, draw GN parallel to AB. With M as a centre, and MG as a radius, the circumference evidently passes through N. It is also clear that CF is perpendicular to AD and passes through G.

Now, $CM = \frac{1}{2}(AC + CB)$, and $ME = \frac{1}{2} AB$.

Hence $CE = \frac{1}{2}(AC + CB + AB) = \frac{1}{2} S$,

S being the sum of the sides; hence, also,

$$HE = AO = \tfrac{1}{2} S - CB, \quad CO = \tfrac{1}{2} S - AB,$$

and $$AE = \tfrac{1}{2} S - AC.$$

Again, $CF \times AF =$ area ACD;

and $GF \times AF =$ area ABD;

subtracting,

$$AF(CF - GF) = AF \times CG = \text{area } ABC. \quad (1)$$

Once more,

$$CG \times HG = \text{area } CHB;$$

and $$GF \times HG = \text{area } AHB.$$

Adding,

$$HG(CG + GF) = NA \times CF = \text{area } ACB. \quad (2)$$

Multiplying (1) and (2), we have

$$NA \times AF \times CG \times CF = (\text{area } ABC)^2.$$

But $CF \times CG = CE \times CO$,

and $NA \times AF = AE \times AO = AE \times HE$.

Therefore, area $ACB = \sqrt{CE \times CO \times AE \times HE}$

$$= \sqrt{\tfrac{1}{2} S(\tfrac{1}{2} S - AB)(\tfrac{1}{2} S - AC)(\tfrac{1}{2} S - CB)}.$$

Gain of Wells's Mathematics

The following verified statistics show the percentage of gain in the adoptions of Wells's Mathematics in several states, since this series was purchased by D. C. Heath & Co. in December, 1898:

	Gain in Adoptions
Ohio	**34%**
Indiana	**252%**
Illinois	**91%**
Wisconsin	**53%**
Michigan	**53%**
Kentucky	**300%**
Tennessee	**147%**
Missouri	**358%**
Iowa	**63%**
Kansas	**275%**
Nebraska	**371%**
Minnesota	**167%**

During this time, so far as known, **not one school has displaced Wells's Mathematics** in this territory.

D. C. HEATH & CO.
Publishers
Boston New York Chicago London

FAMOUS
GEOMETRICAL THEOREMS
AND PROBLEMS

WITH THEIR HISTORY

BY

WILLIAM W. RUPERT, C.E.
SUPERINTENDENT OF SCHOOLS, POTTSTOWN, PA.

> "I am sure that no subject loses more than mathematics by any attempt to dissociate it from its history." — J. W. L. Glaisher.

BOSTON, U.S.A.
D. C. HEATH & CO., PUBLISHERS
1901

COPYRIGHT, 1901,
BY D. C. HEATH & CO.

PREFACE.

THE author, having derived much pleasure and inspiration from the brief historical notes in some of the mathematical text-books that he studied when a student in college, has thought that, by giving the history of a few of the most celebrated geometrical theorems and problems, he might place a "light in the window" which may throw a cheerful ray adown the long and sometimes dusty pathway that leads to geometrical truth.

In the preparation of this little book most valuable assistance has been derived from Florian Cajori's *History of Mathematics*, James Gow's *History of Greek Mathematics*, and G. J. Allman's *Greek Geometry from Thales to Euclid.* It is, however, to W. W. Rourse Ball's remarkably interesting *Short History of Mathematics* that *Famous Geometrical Theorems and Problems* owes the largest debt.

To Professor A. D. Eisenhower, Principal of the Norristown High School, George Q. Sheppard, Professor of Mathematics, Hill School, Pottstown, Pa., Dr. George M. Philips, Principal West Chester State Normal School, and Daniel Carhart, C.E., Dean and Professor of Civil Engineering, Western University of Pennsylvania, who have read this book in manuscript, the author is indebted for valuable suggestions and many kind words of encouragement.

For the excellent and accurate diagrams, and for proof No. V. in Chapter I., my thanks are due to my young friend and former pupil, Luther D. Showalter, C.E.

WILLIAM W. RUPERT.

POTTSTOWN, PA.,
December 23, 1899.

CHAPTER VI.

THE DUPLICATION OF THE CUBE.

51. This is one of the most famous problems of antiquity, and a halo of interest still lingers around it.

By the duplication of the cube is meant finding the edge of a cube which shall have twice the volume of a cube whose edge is given. Thus if a is the length of the side of the given cube, and x that of the required cube, we have $x^3 = 2a^3$. The duplication of the cube, therefore, like the trisection of an angle, involves the solution of a cubic equation; but this is impossible by Euclidean geometry. The higher geometry, however, furnishes an abundance of solutions.

There has been considerable speculation respecting the origin of the duplication problem. The problem was known in ancient times as the Delian problem, in consequence of a legend that the Delians consulted Plato on the subject.

"In one form of the story," says Professor Ball in his *Mathematical Recreations*, "it is asserted that the Athenians in 430 B.C., when suffering from the plague of eruptive typhoid fever, consulted the oracle at Delos as to how they could stop it. Apollo replied that they must double the size of his altar, which was in the form of a cube. To the unlearned suppliants nothing seemed more easy, and a new altar was constructed either having each of its edges double that of the old one (from which it followed that the volume was increased eightfold), or by placing a similar cubic altar next to the old one. Whereupon, according to the legend, the indignant god made the pestilence worse than before, and informed a fresh deputation that it was useless to trifle with him, as his new altar must be a cube, and have a volume exactly double that of his old one."

The Athenians, now suspecting that the problem could not be so simple as it appeared, applied to Plato, who directed them to the geometricians. Since Plato was born 429 B.C., the insertion of his name is an obvious anachronism.

The origin of this problem has also been traced to King Minos, who wished to erect a tomb for his son Glaucus. Minos, being dissatisfied with the dimensions of the tomb (100 feet each way) proposed by his architect, exclaims: "The enclosure is too small for a royal tomb; double it, but fail not in the cubical form."

It has also been suggested that the duplication problem grew out of the fact that the Pythagoreans had found the diagonal of a square to be the side of a square twice as large as that of which it is the diagonal, and they wished to discover a similar law for the cube.

Hippocrates (cir. 420 B.C.) was the earliest mathematician who made any real progress in the solution of the Delian problem. He made a very important advance when he reduced the question to that of finding two means between one straight line (a), and another twice as long ($2a$). If these means are x and y, we have,

$$a : x :: x : y :: y : 2a.$$

From this proposition we have,

$$(1)\quad ay = x^2, \text{ and } (2)\; y^2 = 2ax.$$

From (1)
$$y = \frac{x^2}{a},$$
$$y^2 = \frac{x^4}{a^2}.$$

Substituting this value of y^2 in (2), and reducing, we have $x^3 = 2a^3$.

If, therefore, we can find a method by which to insert the two means, we can solve the famous problem. For more than two thousand years the problem has been presented in this form. Though Hippocrates succeeded in reducing the problem to a new form, he failed to find the two mean proportionals.

52. Archytas, the Greek philosopher, mathematician, general, and statesman, was one of the first to give a solution. He lived about 400 B.C., and was one of the most respected and influential citizens of Tarentum, now Taranto, Italy. This city was at the head of the league of the Greek cities in Italy against Dionysius of Syracuse.

Archytas is said to have been seven times general of the Tarentine forces, and it is affirmed that he was always victorious. His influence among his contemporaries was very great, and he used it with Dionysius on one occasion to save the life of Plato, whose advice in matters of government was distasteful to that despot. Archytas was noted for the attention he paid to the comfort and education of the children of Tarentum, and was universally admired for his virtues. It was he who first applied mathematical principles to practical machines, and he is credited with the construction of a flying bird and other ingenious mechanical toys. Archytas taught that the earth is a sphere rotating round its axis in twenty-four hours! When we recall that this was 400 B.C., and that in the seventeenth century A.D., Galileo was forced to this deeply solemn recantation, "I, Galileo, being in my seventieth year, being a prisoner upon my knees, having before me the Holy Gospel, which I touch with my hands, abjure, curse, and detest the error and the heresy of the movement of the earth," we get a hint of Archytas' intellectual power, and perceive that, in this particular at least, he was not, as were many subsequent thinkers, bound by the chains of superstition. He was drowned in a shipwreck near Tarentum, and his body washed on shore. This tragic end was regarded by the more rigid Pythagoreans as a punishment brought down upon his head by departing from the lines of study laid down by their illustrious master.

The following solution of the duplication of the cube was worked out by Archytas. An examination of this remarkable and beautiful solution will, we believe, convince the reader that the celebrity of Archytas as a mathematician is well founded.

Let there be two given lines, AD, G; it is required to find two mean proportionals to them.

Let a circle $ABDZ$ be described round the greater line AD; and let the line AB, equal G, be inserted in it; and being produced let it meet at

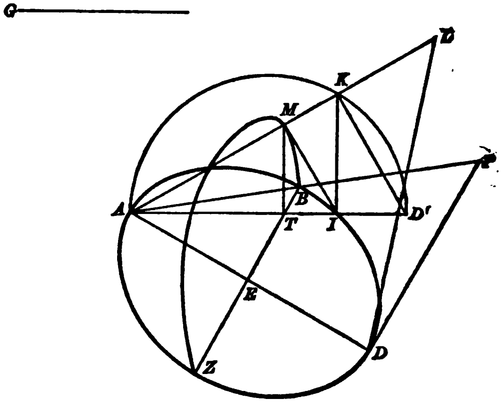


Fig. 51.

the point *P*, the line touching the circle at the point *D*. Further, let *BEZ* be drawn parallel to *PD*.

Now, let it be conceived that a semicylinder is erected on the semicircle *ABD*, at right angles to it; also, at right angles to it, let there be drawn on the line *AD* a semicircle lying in the parallelogram of the cylinder. Then let this semicircle be turned round from the point *D* toward *B*, the extremity *A* of the diameter remaining fixed; it will, in its circuit, cut the cylindrical surface and describe on it a certain line.

Again, if, the line *AD* remaining fixed, the triangle *APD* be turned round, with a motion contrary to that of the semicircle, it will form a conical surface with the straight line AP, which in its circuit will meet the cylindrical line (*i.e.* the line which is described on the cylindrical surface by the motion of the semicircle) in some point; and at the same time the point B will describe a semicircle on the surface of the cone.

Now, at the place of meeting of the lines, let the semicircle in the course of its motion have a position $D'KA$, and the triangle in the course of its opposite motion a position DLA; and let the point of said meeting be K.

Also let the semicircle described by B, be BMZ, and the common section of it and of the circle $BDZA$ be BZ; from the point K let a perpendicular be drawn to the plane of the semicircle BDA; it will fall on the periphery of the circle, because the cylinder stands perpendicularly. Let it fall, and let it be KI; and let the line joining the points I and A meet the line BZ in the point T; and let the right line AL meet the semicircle BMZ in the point M; also let the lines KD', MI, MT be drawn.

Since, then, each of the semicircles $D'KA$, BMZ is at right angles to the underlying plane, and, therefore, their common section MT is at right angles to the plane of the circle; so also is the line MT at right angles to BZ.

Therefore, the rectangle under the lines TB, *TZ*, that is, under *TA*, *TI*, is equal to the square on *MT*.

The triangle *AMI* is therefore similar to each of the triangles *MIT*, *MAT*, and the angle *IMA* is right.

But the angle *D′KA* is also right.

Therefore, the lines *KD′*, *MI* are parallel. And there will be the proportion:

As the line *D′A* is to *AK*, *i.e.* *KA* to *AI*, so is the line *IA* to *AM*, on account of the similarity of the triangles.

The four straight lines *D′A*, *AK*, *AI*, *AM* are, therefore, in continued proportion.

Also the line *AM* is equal to *G*, since it is equal to the line *AB*.

So the two lines *AD*, *G* being given, two mean proportionals have been found, viz. *AK*, *AI*.

Thus runs the solution of the Delian problem by Archytas. If *AD* be taken equal to 2 *G*, we have, $\overline{AI}^3 = 2\,\overline{AM}^3 = \overline{2G}^3$.

The writer of a modern text-book might state the proportions more briefly, and might give the reasons for them more clearly; but we prefer to retain, as nearly as possible, the language of Archytas, which all will understand.

53. The following very elegant solutions of the duplication-problem were given by Manaechmus, who was born about 375 B.C. His solutions, two in number, have been handed down to us by Eutocius in his *Commentary on the Second Book of the Treatise of Archimedes on the Sphere and Cylinder.* Manæchmus acquired great reputation as a teacher of geometry, and was for that reason appointed one of the tutors to Alexander the Great.

In answer to Alexander's requests to make his proofs shorter, he made the well-known reply, "In the country, sire, there are private and even royal roads, but in geometry there is only one road for all."

FIRST SOLUTION BY MANÆCHMUS.

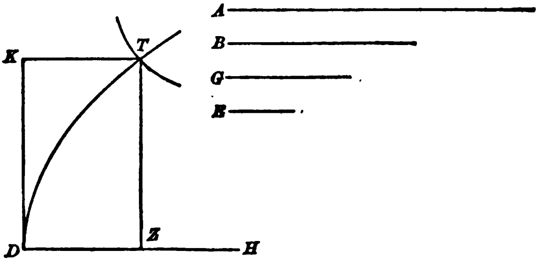


Fig. 52.

Let the two given lines be A, E; it is required to find two mean proportionals between them.

Let it be done, and let them be B, G; and let the straight line DH, given in position and limited in D, be laid down; and at D let DZ, equal to the straight line G, be placed on it, and let the line TZ

be drawn at right angles, and let *ZT*, equal to the line *B*, be laid down.

Since, then, the three straight lines *A*, *B*, *G*, are proportional, the rectangle under the lines *A*, *G*, is equal to the square on *B*.

Therefore the rectangle under the given line *A*, and the line *G*, that is the line *DZ*, is equal to the square on the line *B*, that is to the square on the line *ZT*.

Therefore the point *T* lies on a parabola described through *D*.

Let the parallel straight lines *TK*, *DK* (parallel respectively to *DZ* and *ZT*) be drawn; since the rectangle under *B*, *G* is given (for it is equal to the rectangle under *A*, *E*), the rectangle *KTZ* (*KTZD*) is also given.

The point *T*, therefore, lies on a hyperbola described with the straight lines *KD*, *DZ* as asymptotes.

The point *T* is therefore given; so also is the point *Z*.

The synthesis will be as follows:

Let the given straight lines be *A*, *E*, and let the line *DH* be given in position and terminated at *D*; through *D* let a parabola be described whose axis is *DH* and parameter *A*. And let the squares of the ordinates drawn at right angles to *DH* be equal to the rectangles applied to *A*, and having for breadths the lines cut off by them to the point *D*. Let the parabola be described, and let it be *DT*, and let the line *DK* be drawn and let it be a perpendicular; and with the straight lines *KD*, *DZ* as asymptotes, let the hyperbola be described, so that the lines drawn from it parallel to the lines *KD* and *DZ* shall form an area equal to the rectangle under *A*, *E*; the hyperbola will cut the parabola; let them cut in *T*, and let perpendiculars *TK*, *TZ*, be drawn.

Since, then, the square on *ZT* is equal to the rectangle under *A* and *DZ*, there will be: as the line *A* is to *ZT*, so is the line *ZT* to *ZD*.

Again, since the rectangle under *A* and *E* is equal to the rectangle *TZD*, there will be: as the line *A* is to the line *ZT*, so is the line *ZD* to the line *E*.

But the line *A* is to the line *ZT*, as the line *ZT* is to *ZD*.

And, therefore, as the line *A* is to the line *ZT*, so is the line *ZT* to *ZD*, and the line *ZD* to *E*.

Let the line *B* be taken equal to the line *ZT*, and the line *G* equal to the line *DZ*.

There will be, therefore, as the line *A* is to the line *B*, so is the line *B* to the line *G*, and the line *G* to *E*.

The lines *A*, *B*, *G*, *E*, are therefore, in continual proportion; which was required to be found.

If we consider the above solution in the light of modern analysis we have a good illustration of the beauty and power of this analysis. For, describe a parabola of latus rectum l, and also a rectangular hyperbola with $4l$ for its real axis, and having for its asymptotes the tangent at the vertex of the parabola and the axis of the parabola. Then the ordinate and the abscissa of the point of intersection of these curves are the mean proportionals between l and $2l$. For we know the equation of the parabola is $x^2 = ly$, and that of the hyperbola is $xy = 2l^2$.

Solving these equations for x and y, we have

$$x^3 = 2l^3 \text{ and } y^3 = 4l^3,$$

and thus the point in which the curves cut is determined.

Hence, also, $l:x::x:y::y:2l$.

SECOND SOLUTION BY MANÆCHMUS.

54. Let AB, BG be the two given straight lines placed at right angles to each other; and let their mean proportionals be DB, BE, so that, as the line GB is to BD, so is the line BD to BE, and the line BE to BA, and let the perpendiculars DZ, EZ be drawn.

Since, then, there is: as the line GB is to BD, so is the line BD to BE; therefore the rectangle

GBE, that is, the rectangle under the given straight line, *GB*, and the line *BE* will be equal to the square on *BD*, that is the square on *EZ*.

Since, then, the rectangle under a given line and the line *BE* is equal to the square on *EZ*, therefore the point *Z* lies on a parabola described about the axis *BE*.

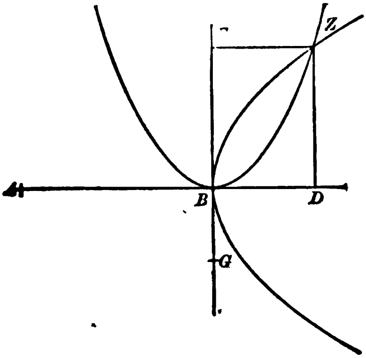


Fig. 53.

Again, since there is: as the line *AB* is to *BE*, so is the line *BE* to *BD*; therefore the rectangle *ABD*, that is, the rectangle under the given straight line, *AB*, and the line *BD*, is equal to the square on *EB*, that is the square on *DZ*.

The point *Z*, therefore, lies on a parabola described about the axis *BD*.

But the point *Z* lies also on another given parabola described about the axis *BE*. The point *Z* is therefore given; as also the perpendiculars *ZD*, *ZE*.

The points *D*, *E*, are, therefore, given.

The synthesis will be as follows:

Let *AB*, *BG* be the two given lines placed at right angles to each other, and let them be produced indefinitely from the point *B*; and let there be described about the axis *BE* a parabola, so that the square on any ordinate, *ZE*, shall be equal to the rectangle applied to the line *BG* with the line *BE* as height.

Again, let a parabola be described about *DB* as axis, so that the squares on its ordinates shall be equal to rectangles applied to the line *AB*. These parabolas cut each other; let them cut at the point *Z*, and from *Z* let the perpendiculars *ZD*, *ZE* be drawn.

Since, then, in the parabola, the line *ZE*, that is, the line *DB*, has been drawn, there will be: the rectangle under *GB*, *BE* equals the square on *BD*.

There is, therefore, as the line *GB* is to *BD*, so is the line *DB* to *BE*. Again, since in the parabola the line *ZD*, that is, the line *EB*, has been drawn, there will be: the rectangle under *DB*, *BA* equals the square on *EB;* there is, therefore, as the line *DB* is to *BE*, so is the line *BE* to *BA*.

But there was, as the line *DB* is to *BE*, so is the line *GB* to *BD*.

And thus there will be, therefore, as the line GB is to BD, so is the line DB to BE, and the line BE to BA; which was required to be found.

Analysis makes short work of this solution. To show, analytically, that the construction by Manæchmus is correct, let the equations of the parabolas be $y^2 = 2ax$ and $x^2 = ay$. They intersect in a point whose abscissa is given by $x^3 = 2a^3$.

55. We shall now conclude this chapter with a very simple and beautiful solution of the Delian problem which is effected by means of the cissoid or "ivy-like" curve. This curve was invented by the Greek geometrician Diocles, and it is generally referred to as the cissoid of Diocles. This mathematician lived about 180 B.C., and the cissoid was invented by him for the express purpose of solving the problem now under discussion. Though Diocles was one of the distinguished mathematicians of his day, very little is known concerning either him or his work. It is the cissoid that has kept his memory green.

For the substance of the following proof I am indebted to Edward Olney, who gives it in one of his excellent works, *General Geometry and Calculus*, which the author studied with pleasure more than twenty years ago.

An examination of the following definition and diagram will give a clear conception of the form of the cissoid of Diocles. The curve is defined as follows: "If pairs of equal ordinates be drawn to the diameter of a circle, and through one extremity

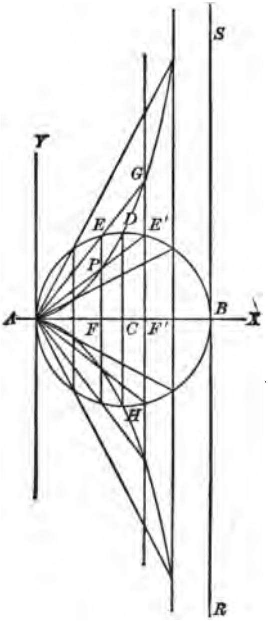


Fig. 54.

of this diameter and the point in the circumference through which one of the ordinates is let fall, a line be drawn, the locus of the intersection of this line and the equal ordinate, or that ordinate produced, is the cissoid of Diocles."

Let AB be the diameter of a circle; and let EF and $E'F'$ be equal ordinates.

Through A and E' draw AE' intersecting EF in P.

Then is P a point in the curve.

In like manner draw AE and produce it till it meets $E'F'$ produced in G.

Then is G also a point in the curve.

In like manner other points are determined and the curve is traced through them.

It will be seen that the cissoid has two branches as shown in the figure. These branches meet at A, pass through the extremities of the diameter DH, and have RS as a common asymptote.

We are now ready to produce the equation of the curve.

Let AX and AY be the axes of reference, $AB = 2a$, the diameter of the circle referred to in the definition, and P any point in the curve.

Then $AF = x$, and $PF = y$.

It will be remembered that APE' is, by definition, a straight line.

We now have, by similiar triangles,

$$AF : PF :: AF' : E'F', \quad x : y :: 2a - x : \sqrt{(2a-x)x},$$

since $E'F'$ is a mean proportional between the segments of the diameter made by that ordinate.

Squaring and reducing, $x^2 : y^2 :: 2a - x : x$.

$$\therefore \quad y^2 = \frac{x^3}{2a - x}.$$

This, therefore, is the general equation of the cissoid of Diocles.

56. Now that we have the equation of the graceful, ivy-like cissoid, we are prepared to show how beautifully it disposes of the celebrated Delian problem.

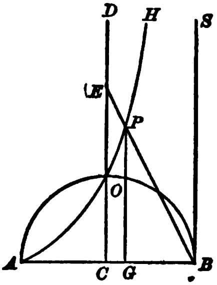


Fig. 55.

Let AH be any cissoid, AB the diameter of the circle which belongs to it, and C the centre of that circle.

Take $EC = 2\,CB$ and draw EB.

From the point P, where EB cuts the cissoid, let fall the perpendicular PG. Since $EC = 2\,CB$, $PG = 2\,GB$.

Now a cube described on PG is twice one described on AG; for since $PG = y$, $AG = x$, and $GB = 2a - x$, we have

$$\overline{PG}^2 = \frac{\overline{AG}^3}{GB} = \frac{\overline{AG}^3}{\frac{1}{2}PG},$$

or $$\tfrac{1}{2}\overline{PG}^3 = \overline{AG}^3.$$

Hence $$\overline{PG}^3 = 2\,\overline{AG}^3.$$

Finally, let e be the edge of *any* given cube; find e_1, so that $e : e_1 :: AG : PG$,

whence $$e^3 : e_1{}^3 :: \overline{AG}^3 : \overline{PG}^3.$$

But $$\overline{PG}^3 = 2\,\overline{AG}^3.$$

Hence $$e_1{}^3 = 2\,e^3.$$

It is easy to show that $\overline{OG}^3 = 2\,\overline{GB}^3$; also that $\overline{AG}^3 = 2\,\overline{OG}^3$.

For, from the equation of the cissoid, we have,

$$\overline{PG}^2 = \frac{\overline{AG}^3}{GB},$$

and by elementary geometry,

$$\overline{OG}^2 = AG \times BG.$$

$$\therefore \frac{\overline{PG}^2}{\overline{OG}^2} = \frac{\overline{AG}^3}{AG \times \overline{GB}^2} = \frac{\overline{AG}^2}{\overline{GB}^2}.$$

Now $$\frac{\overline{PG}^2}{\overline{OG}^2} = \frac{\overline{AG}^2}{\overline{GB}^2} \text{ or } \frac{\overline{PG}^3}{\overline{OG}^3} = \frac{\overline{AG}^3}{\overline{GB}^3}.$$

Putting this into the form of a proportion we have,

$$\overline{PG}^3 : \overline{OG}^3 :: \overline{AG}^3 : \overline{GB}^3,$$

and, by alternation,

$$\overline{PG}^3 : \overline{AG}^3 :: \overline{OG}^3 : \overline{GB}^3.$$

$$\therefore \quad \overline{PG}^3 : 2\,\overline{AG}^3 :: \overline{OG}^3 : 2\,\overline{GB}^3.$$

But $\quad \overline{PG}^3 = 2\,\overline{AG}^3.$

$$\therefore \quad \overline{OG}^3 = 2\,\overline{GB}^3.$$

It is now plain that $\overline{AG}^3 = 2\,\overline{OG}^3$.

By taking $EC = 3\,CB$ and proceeding in a similar manner, we can triplicate the cube; or, indeed, in the same way obtain the edge of a cube of any given number of times the volume of a given cube.

We have all heard of "good measure, pressed down, heaped up, and running over." The duplication of the cube by means of the cissoid of Diocles might appropriately be called "good duplication, pressed down, heaped up, and running over," for the ivy-like curve really gives three different lines whose cubes are twice the cubes of other lines. Then, too, if any king or other potentate wants his cubical altar triplicated, or, indeed, augmented in any integral ratio whatever, the cissoid will be the refuge of all who may wish to comply with the royal commands.

There are many other solutions of this celebrated problem, but it would consume too much space to give all. I have confined myself to the solutions having the most historic interest. These, too, as it happens, are the most famous and beautiful.

CHAPTER VII.

MATHEMATICAL INSCRIPTION UPON THE TOMBSTONE OF LUDOLPH VAN CEULEN.

57. Ludolph van Ceulen (1539–1610) devoted his life to finding a numerical approximation to the value of π. Before death overtook him, in 1610, he succeeded in making his approximation correct to 35 places of decimals. In 1596 he had the result worked out to 20 places, and this value was calculated by using regular inscribed and circumscribed polygons of 60×2^{33} sides. It will be seen that the last 15 places cost him fourteen years' work. Just how many sides Van Ceulen's polygons had, at the time of his death, I do not know; but, according to Professor Ball, they must have been above 2^{62}, *i.e.* 4,611686,018427,387904 in number. Polygons with 2^{62} sides give a result correct to 32 places.

It should be remembered that Van Ceulen did not have the aid of logarithms in his day, and that his calculations were necessarily made in a very laborious manner. These calculations, indeed, were considered so remarkable that the value of π has, in his honor, been called "Ludolph's number"; and his approximation to the value of π (3.141,592,653,589,793,238,462,643,383,279,502,88) was, in compliance with his request, carved on his tombstone in St. Peter's churchyard at Leyden.

"Ludolph's number" has been carried now to more than 700 places; but the greater accuracy has been attained by using modern methods which are as far superior to those at Van Ceulen's command as steam and electricity are superior to human muscle.

FAMOUS MATHEMATICAL DEVICES ON TOMBSTONES.

58. James Bernoulli, a distinguished mathematician, born at Basel, Switzerland, December 27, 1654, when investigating the properties of the equiangular spiral, noticed the manner in which various curves deduced from it reproduced the original curve. This fact (an unusual one) strongly impressed the mind of the great mathematician, and he requested that an equiangular spiral should be engraved on his tombstone with the inscription "eadem numero mutata resurgo."

Professor Ball, speaking of James Bernoulli, says: "He was one of the earliest to realize how

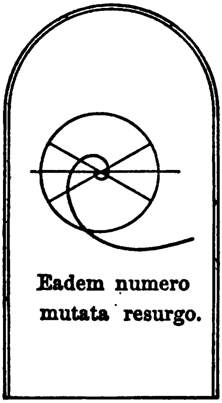


Fig. 56.

powerful as an instrument of analysis was the infinitesimal calculus. His lectures, which contain the earliest use of the term integral, were the first published attempt to construct an integral calculus; for Leibnitz had treated each problem by itself, and had not laid down any general rules on the subject."

Before passing to the consideration of the next device, we may remark that the equiangular spiral is so called because the angle included between any radius vector and the tangent at the extremity is constant. This spiral is also frequently called the logarithmic spiral because the radii vectors making equal angles with each other are in geometrical progression. It is evident, then, that the measuring arcs, being in arithmetical progression, are as the logarithms of the radii vectors which are in geometrical progression.

It will thus be seen that either name indicates a characteristic of the spiral that adorns the stone marking the resting-place of James Bernoulli.

59. We have now reached the celebrated geometrical device carved on the tomb of the Newton of antiquity, Archimedes. But before considering the device, let us pause a moment to contemplate the man and his work.

Archimedes, the greatest mathematician the world has yet seen, excepting Newton only, was born in Syracuse about 287 B.C. Plutarch says he was a relative of King Hiero of Syracuse, but we shall probably follow a safer guide if we believe Cicero, who says he was of humble birth. Nor does Cicero stand alone in this; for Silius Italicus, speaking of Archimedes, says, "Nudus opum, sed cui cælum terræque paterent." If Cicero and Italicus are right in this matter, we have another important fact to give plausibility to the induction, "Great men are like mountains; on one side is the valley of ancestors, on the other the depression of posterity."

Some of the works of Archimedes have been lost, but the following books have come down to us: *Equiponderance of Planes*, *Quadrature of the Parabola*, *The Sphere and Cylinder*, *Measurement of the Circle*, *Spirals*, *Conoids and Spheroids*, *The Sand-Counter*, and *Floating Bodies*.

Archimedes' proof of the mathematical principle of the lever, given in the *Equiponderance of Planes*, holds its place in the text-books of to-day. He was, indeed, the author of the first sound knowledge of mechanics, and for eighteen hundred years no one was even able to follow the path which he opened. This important branch of mathematics remained absolutely where Archimedes left it until Stevinus, in 1586, published his treatise on statics.

The Sphere and Cylinder contains sixty propositions, and it was in the demonstrations found here that Archimedes himself took most pride.

The *Measurement of the Circle* contains but three propositions, but they are of great value and beauty. In the first proposition Archimedes proves that the area of a circle $= \pi r^2$. In the second proposition he shows that $\pi r^2 : (2\,r)^2 :: 11 : 14$ very nearly. In the third proposition he shows that π is less than $3\frac{1}{7}$ and greater than $3\frac{10}{71}$.

In *The Sand-Counter* Archimedes says men talk of the sand of the Sicilian shore as being absolutely beyond the power of calculation. But he says he will teach them better; and he then proceeds, not to calculate the sand grains on the Sicilian shore, but to find a superior limit to the number of grains of sand which would fill the whole universe. We may add that in the language of Archimedes the "whole universe" referred to a sphere whose centre is the earth, and radius the distance of the sun.

Floating Bodies contains nineteen propositions in two books. There is an interesting story told of the manner in which the attention of the great mathematician was directed to hydrostatics. Hiero, the king of Syracuse, desiring a crown of gold, gave the precious metal to a goldsmith with instructions for making the same. In due time the crown came to the king's hand. It was of the proper weight; but, for some reason, it was suspected that the goldsmith had put some of the gold into his own pocket, replacing it with an equal weight of silver. Hiero in his trouble appealed to Archimedes. Even the great Archimedes was puzzled. He did not at once see his way to a solution, but one day not long after, when in the public baths, he noticed that his body was pressed upward by a force that grew greater and greater as he sank farther and farther into the water. This was the hint seized by the great mind. He rushed from the bath, just as he was, and ran home through the streets, shouting *εὕρηκα, εὕρηκα,* "I have found it, I have found it." He at once ascertained that when equal weights of gold and silver were weighed in water, they no longer appeared equal; each, by the weight of water it displaced, seemed lighter than when weighed in air; but as the silver was more bulky than gold, its apparent weight suffered the greater change. If, then, on one scale-pan of a balance he placed the crown, and on the other an equal weight of pure gold, and then immersed the whole in water, the gold would outweigh the crown if any silver had been substituted for the gold. Tradition says that the scale-pan containing the crown went up, and thus the goldsmith was shown to be dishonest.

"After examining the writings of Archimedes," says Cajori, "one can well understand how, in ancient times, an 'Archimedean problem' came to mean a problem too deep for ordinary minds to solve, and how an 'Archimedean proof' came to be the synonym for unquestionable certainty."

Archimedes, like Plato, held the peculiar belief that a philosopher should not seek to apply the results of science to practical affairs. "Every kind of art," said he, "which is connected with daily needs is ignoble and vulgar." Certain it is, however, his practice was not in complete harmony with this theory, for he actually did turn his scientific knowledge to eminently practical purposes. Upon one occasion King Hiero was unable to launch an unusually large ship which had been built under his directions. Archimedes was called, and by means of cogwheels and an endless screw he soon had the ship afloat. It was upon this occasion, we are told, that Archimedes, when responding to the compliments of the king, declared he could move the whole world if he only had a place whereon to rest his fulcrum. The Archimedean screw was used in Egypt to drain the fields after they had been flooded by the Nile, and it was also used in pumping water from the holds of ships. At Syracuse the Romans found it hard enough to fight against his powerful catapults; and his wonderful burning glasses, even if it should be granted they were not strong enough to fire the Roman fleet, were a very practical application of science.

But at last neither the wisdom of Archimedes nor the bravery of the citizens of Syracuse availed against the Romans. Syracuse fell, and, sad to tell, in the indiscriminate slaughter which followed the great Archimedes was killed. This happened, however, through no fault of Marcellus, who commanded the Romans, for he gave strict orders that the house and the life of the great mathematician should be spared. It is said that a sturdy Roman soldier entered the study of Archimedes while he was pondering over a geometrical diagram drawn, as was the custom of that day, in the sand. Archimedes commanded him to get off his diagram, and not spoil it. The soldier, feeling insulted because a citizen of a captured city gave him orders, and not knowing who he was, killed him with his spear.

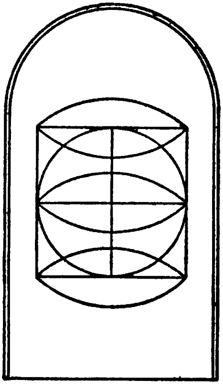


Fig. 57.

The great geometrician, in his favorite work, *The Sphere and Cylinder*, proved that the surface of a sphere is to the entire surface of the circumscribed cylinder, including its bases, as 2 to 3; and their volumes are in the same ratio. Archimedes thought so much of this theorem, and of those leading up to it, that he expressed a wish that the device on his tombstone should be a sphere inscribed in a cylinder. Marcellus, let it be remembered to his honor, buried the philosopher with honors, and on a splendid tomb he executed the device of the sphere and cylinder.

Cicero gives (*Tusc. Disput.*, V. 23) an account of the manner in which he discovered this tomb in 75 B.C. We close the sketch with C. D. Yonge's translation of Cicero's charming narrative:

"Shall I not, then, prefer the life of Plato and Archytas, manifestly wise and learned men, to his (Dionysius), than which nothing can possibly be more horrid, or miserable, or detestable?

"I will present you with an humble and obscure mathematician of the same city, called Archimedes, who lived many years after: whose tomb, overgrown with shrubs and briers, I in my quæstorship discovered, when the Syracusans knew nothing of it, and even denied that there was any such thing remaining; for I remembered some verses which I had been informed were engraved on his monument, and these set forth that on the top of the tomb there was placed a sphere with a cylinder. When I had carefully examined all the monuments (for there are a great many tombs at the gate Achradmæ), I observed a small column standing out a little above the briers, with the figure of a sphere and cylinder upon it; whereupon I immediately said to the Syracusans — for there were some of their principal men with me there — that I imagined that was what I was inquiring for. Several men, being sent with scythes, cleared the way, and made an opening for us. When we could get at it, and were come near to the front of the pedestal, I found the inscription, though the latter parts of all the verses were effaced almost half way.

"Thus one of the noblest cities of Greece, and one which at one time likewise had been very celebrated for learning, had known nothing of the monument of its greatest genius, if it had not been discovered to them by a native of Arpinum."

THREE (3) POINTS

— FROM MANY — IN WHICH

WELLS'S ESSENTIALS OF GEOMETRY

EXCELS:

Accuracy

No other Geometry is so free from ambiguous and loosely constructed statements. Every definition and demonstration has been subjected to rigorous criticism in order to secure clearness, brevity and absolute accuracy.

Adaptation

to the needs of beginners. The difficulties that confront the pupil are recognized and met in such a way as to arouse his interest and enthusiasm. Propositions and original exercises are presented in a manner at once more teachable and more educative than ever before attempted.

Adequacy

to the demands of the colleges and technical schools. The entrance requirements are heeded, *both in letter and spirit,* without sacrifice of organic unity.

Summary

1. **Accurate for Everybody**
2. **Interesting to Pupils**
3. **Satisfactory to Teachers**

No teacher in search of the best and most practical text on Geometry can afford to disregard the merits of Wells's Essentials.

D. C. HEATH & CO., Publishers
BOSTON NEW YORK CHICAGO

2/

ON TEACHING GEOMETRY

BY

FLORENCE MILNER
DETROIT UNIVERSITY SCHOOL
DETROIT, MICHIGAN

BOSTON, U.S.A.
D. C. HEATH & CO., PUBLISHERS
1900

COPYRIGHT, 1900,
BY D. C. HEATH & CO.

ON TEACHING GEOMETRY.

Of writing many geometries there is no end. With any of them, or without them all, the good teacher will get good results; with the best of them, the poor teacher cannot rise above mediocrity. Under both conditions, however, there is wisdom in a careful choice, for a strong book not only lessens the labors of a good teacher, but makes it possible for a class to get some value out of the work in spite of poor teaching. Yet we as teachers are inclined to ask too much of text-books, and we expect them not only to do their own work, but also to become responsible for a large share of ours. It is the province of the text-book to present clearly, according to its established sequence, the subject matter of geometry, not to teach how to teach it, and that book is best which has in it least of anybody's method, even of the present writer's, and most of clearly expressed geometry. So, however good the book, there always remains, and wisely too, much for the teacher to do.

It is not the present purpose to outline any particular method of teaching, but to call attention to a few general lines which, in harmony with the subject, should be followed out, and to touch upon some points in a wider training which a correct handling of the subject should give.

Geometry differs markedly from the preceding studies, and success or failure depends largely upon getting at the beginning the right point of view. To play a game of chess one must learn the moves of individual pieces. These, in infinite combination, but always under the same fixed rules, make up the most intricate of intellectual games. Geometry is analogous. We expect no one to play a game of chess until he learns the moves; we should not expect a pupil to work intelligently in geometry until he is helped to the mental attitude demanded by the subject, and knows something of the few simple truths that are the guiding thread through the seemingly intricate labyrinth.

In geometry the pupil encounters for the first time formal logic, and we are too prone to plunge a class into it without adequate preparation. Before opening the book at all, the pupil should be taught something of the nature of logic, something of its requisites, something of its method of work. Certain texts adhere to set, formal demonstrations, others give nothing but original work, while between these two extremes are all grades of compromise, with more or less practical application of established truths. Whether we are working out original theorems or are following the demonstration of a conventional proposition or are using our knowledge in the solution of a practical problem, the process is identical.

Axioms and definitions are the foundation upon which the whole superstructure of geometry is builded, and in the beginning a class should be thoroughly grounded in an understanding of their nature and province. Definitions should receive first attention.

If you and I are to carry on a discussion about an apple, we must agree as to its characteristics. If you define it as "the fleshy pome or fruit of a rosaceous tree," while I insist that it is a hard, black mineral, we shall never get far in our argument. If you say that "a triangle is a plane figure bounded by three right lines," while I call a solid bounded by four curved surfaces a triangle, our discussion will again come to grief. It becomes necessary, then, that you and I agree upon a common conception of the object under consideration, and that we shall so describe it that ambiguity is impossible. This is the province of the definition. A definition is such a description as will bring up to all minds the same conception. The definitions in geometry have long been agreed upon by mathematicians, and the importance of knowing, and knowing with verbal accuracy, the universally accepted description of geometric concepts cannot be too strenuously insisted upon, and the teacher who in place of accurate technical language accepts careless and verbose statements, does a class a great wrong. A lover of literature will have little patience with one who, pretending to give Hamlet's soliloquy or Milton's sonnet *On his Blindness* would dare substitute his own meagre words for the matchless language of the great masters. In certain directions the phraseology of mathematics has crystallized equally with other literature, and the giving of familiar definitions is not the place for original work,—that comes later.

The attitude toward axioms should next be made right. Every sane mind finds itself unconsciously in possession of certain knowledge. We know that certain things are true. No one can remember when he learned that the whole of anything is greater than any one of its parts, but the babe creeping along the floor has a practical knowledge of the truth, and he will contest it with you to the limit of his physical strength. Later he will formulate the idea, and possibly when he gets into the high school, some teacher will set him to learning this axiom as though it were something new. On the contrary, the youth should be taught that certain facts are so evidently true that everybody must needs accept them or be counted of unsound mind. When he begins the study of geometry, many of these truths should be put into compact form for future use. Here again comes the value of systematic and even stereotyped phraseology. He should learn now to state in the technical language of mathematics what he and everybody else have long known.

When the pupil has been put in the right attitude toward axioms and definitions, he is ready to open his geometry and learn how to use them. Certain general definitions should be studied. He should know what a proposition is; that there are various kinds, differing in purpose and in form of expression. As he comes to them, he should discriminate carefully between theorem, problem, corollary, and lemma.

Most geometries give some preliminary work consisting largely of definitions and other discussion of geometrical concepts. This should be studied with more or less thoroughness, as the preparation of the class demands.

Entering upon the work peculiar to geometry, we come to the opening section, which may treat of perpendicular straight lines, of triangles, or of whatever the sequence of the particular text in use demands. After reading the first theorem, the hypothesis should be isolated that the pupil may know exactly what is to be his by gift of this same hypothesis, and he should be taught to enumerate these gifts. Perhaps the enumeration will run like this: "A line, a point without that line, and a perpendicular from the given point to the given line." Show the class that it is not sufficient to say that a line and a point are given; this allows one to locate the point within the line. Whenever they fail to be thus accurate in statement, draw figures, taking such latitude as they leave you by inaccurate statement. A few illustrations will show them the necessity of talking sharply to the facts, and the pupils will soon learn to hold each other to statements that admit of no ambiguity.

A certain power of reconstructive imagination should also be cultivated. Besides stating accurately the conception that is in one's own mind, there must be the ability to construct, rapidly and clearly, mental pictures of all statements made in class. Every sentence uttered should add something to the picture or present some new phase of it. Every recitation should be a series of constantly changing views in which everybody, in his mind's eye, sees the same things. This result is not easy to obtain, but it is possible, and the degree of excellence reached by a class becomes a crucial test of the teacher.

Impress upon a class that the formal statement of a proposition is always a *general* truth, true of all figures falling under the given conditions. These statements are conventional and should be as accurately stated as definitions and axioms.

In proving a proposition, our human limitations require us to fix our minds upon a particular case, and accordingly a *special* statement follows.

Given 1. Line AB.
2. Point P without the line.
3. Perpendicular PD upon the line

To prove, that *PD* is the shortest distance from *P* to the line *AB*.

If there is more than one conclusion, bring out the facts in clear, mathematical language. Do not be satisfied to let a class say, "To prove that from the point *P* one perpendicular and only one can be let fall to the line *AB*." It should be stated, "To prove: first, that one perpendicular can be let fall upon the line *AB*; and second, that only one can be let fall from the same point upon the line."

This may seem to some like splitting hairs, but mathematics is an exact science, and in the beginning too much cannot be done in training toward accurate thought and exact expression.

Now comes important work in preparing for a formal demonstration. Here the class must learn that the regular, logical form of every argument is a syllogism. Pupils will not be frightened at the new word, and they will like to use it when they comprehend its meaning. Discuss it with them. Explain the major premise, the minor premise, and show how the conclusion inevitably follows. Give them an example and set them to hunting for others. You will wonder where they find so many.

Next teach them that every step in a well-constructed demonstration is taken by means of a syllogism. As the major premise must always enunciate a general truth, the major premise in their first demonstration must be either an axiom or a definition, for these are the only general geometric truths in their possession. The minor premise isolates and states formally the special case now present and in harmony with the major premise. The conclusion is the new knowledge revealed by bringing these two premises together. This conclusion in turn may become the minor premise, and so on to the end of the argument.

Show a class how a syllogism may be worked out from some of the early definitions and axioms, letting the minor premise be a fact given. Tell them, for instance, to draw the line *CD* to the point *D* in the line *AB*, making the angle *CDB* a right angle. This is the fact given, and if demonstration were to follow, might constitute the hypothesis. Now if the fact is of any value in constructing a syllogism it must fall under some general definition or axiom. Some one will discover that the definition for a perpendicular fits the case. Then you can show how the inevitable conclusion that *CD* is perpendicular to *AB* must follow. Finally show the formal structure of a syllogism and write out the one developed.

1. The line *CD* meets the line *AB*, making the angle *CDB* a right angle.
2. A perpendicular to a line is a line which makes right angles with a given line.
3. *CD* is perpendicular to *AB*.

Repeat the process until you make them see that when the first two terms are rightly selected and agreed upon, there is no escape from the conclusion. It follows like a decree of inexorable fate.

Every demonstration is a chain of syllogisms, in all of which the major premise must be some general conclusion already established, either axiom, corollary, theorem, definition, or algebraic truth. Of course as the work goes on, the formal statement that involves frequent repetition can be contracted, but the habit of looking at every demonstration from this side is invaluable.

Text-books do not follow this form, for they rarely give complete demonstrations. The text is an outline of the subject, a note-book, keeping the general trend of the argument but leaving the pupil to fill in the suggested discussion. Teachers rarely insist upon so closely logical a demonstration as is here outlined, but experience and careful comparison have convinced me that the syllogistic method closely adhered to in the beginning produces results that come from no other kind of work.

In the illustration just used, a pupil might say and with truth, "CD is perpendicular to AB because CDB is a right angle," and "When a line meets," etc., but that is making a statement and then going back to try to make the statement good. The other method never leaves a chance to question a point, for one follows another in the logical order that carries conviction at every step.

If you announce to an opponent the thing of which you expect to convince him, the effect is usually to arouse his antagonism, and he will then and there make up his mind, and no argument of yours, no matter how convincing, is likely to change his opinion. He may listen to you politely, but when you are through he will probably say, "Yes, but as I said in the first place." Don't give him a chance to say anything in the first place, but make him grant your premises, one after another, then when your conclusion is reached, there is nothing left for him to do but accept it.

If this method is used in conventional and formal demonstration, it becomes of greatest value in original work. A pupil thus trained will, when given original work, have a definite and efficient method of attack. He will first study his hypothesis, isolating each individual part of it. Taking one fact for a minor premise, he will next examine his stock in trade and see what definition, theorem, corollary, or other authority he can bring to bear on the case in hand. He will draw his conclusion and see if it advances his argument or brings him nearer the desired end. If it does not, then he decides that his minor premise is wrong and returns to his hypothesis for another fact, and constructs the syllogism. Sooner or later he is bound to reach the right conclusion.

Sometimes teachers allow pupils to say that such or such a thing is true "by a previous proposition." This is a much abused and greatly overworked expression. It becomes to a poor student like a cloak of charity, and if permitted, will be used to cover a multitude of hazy, illogical, and vague ideas.

Do you think that a judge in court would admit as evidence the statement that somewhere, sometime, somebody had made a decision in a case somewhat similar to the one in hand? Indeed not. The lawyer would have to produce his authority, giving title, volume, and page of report, with date of decision. In all demonstrations equal accuracy should be demanded. Not that chapter and verse be given, but pupils should be made to quote geometric scripture in proof of every conclusion. And "quote" here means exactly what the word indicates, not a slipshod attempt at giving the idea.

There are certain definite and wide-reaching results that should come from the right teaching of geometry. Mention has already been made of the importance of accurate and exact expression, but this must be preceded by equally exact and accurate thinking. Better than any other subject, geometry will train the youth to keep close watch and ward over the action of his mind and accustom him to express clearly and honestly the result of his own mental happenings. The ability to make wise selection under varying circumstances, is repeatedly demanded. After a figure is drawn, the pupil should examine it carefully. He may discover in it many things which he knows, from construction, hypothesis, or other conditions, are all true, but only one of them has any concern with the business in hand. He should be trained to see that one fact, and be able to make proper use of it. Teach him to go straight after the one that he needs, and make it serve him.

He should also be taught that certain things lie within his power to do, while others are as absolutely beyond his control as are the turbulent waves, the floating clouds, or the sweeping course of the planets.

He may push a stone from the edge of an overhanging cliff, but after that he must let it go crashing down the mountain side. He may elect to let fall a perpendicular from a certain point to a given line, but he cannot dictate where it shall strike the line. He may know certain things about two figures, and it may be wise to try the application of one to the other. He may lift one figure and place a line of it or an angle upon its equal, but there his control over the matter ceases. From that moment he is under law, and face to face with *the eternal*. Let him stand there a humble spectator, knowing that till heaven and earth shall pass, "one jot nor one tittle shall in nowise pass from the law" over which his finite mind has no control. He was responsible for placing the figures together, but before the resulting consequences he is helpless. He may watch to see if in the finality there is anything that concerns him. Here again he will discover various things accomplished; but again he must recognize the one conclusion for which he has been striving, must isolate it from the rest, and hold it up to view with the strength of conviction.

The young mind is naturally unreasoning, and often utters words without consciousness of a definite idea back of them. The pupil will watch the teacher rather than himself, to determine whether he is travelling the right road. It is very easy for a teacher unconsciously to take this responsibility, and by various gentle leading-strings keep the pupil in the path; but such work is not teaching geometry. If a class leans upon you, or has the habit of watching you, rest assured that your teaching is not right. If the development is correctly carried on, the pupil will be aroused to watch his own mind, his own statements, and the work in hand; the more nearly he can forget *you*, the better. He must learn to discover what is actually happening in his own mind as the process of reasoning goes on; he must be trained to faith in his own convictions, and to fearlessness in expressing them. Nothing is more pitiful than a mind that can be shifted from any position by an incredulous question. At first your pupils will not think independently. Rouse them from this condition, and force them to a conclusion of some sort. A wrong opinion is better than no opinion at all, for activity is better than stagnation.

In a demonstration the teacher must be constantly in the attitude of the doubting Thomas, who took nothing on faith, but always demanded evidence. In this way the burden of proof rests with the pupil, and his aim should be to anticipate every possible question. An exceptionally good teacher of geometry says that the right results have not been attained until a pupil not only ceases to lean upon a teacher, but is also able to stand up against a teacher. Do not rest until you see your pupils thinking independently, and at the same time clearly, fearlessly talking out their conclusions.

For a complete setting in order of one's mental house, the High School course offers nothing better than geometry. The facts learned have some value, but the greatest good is the mental poise, the clearness of vision, and the honesty of expression that it develops. In addition to all this, a pupil rightly trained will learn to measure his own strength, will recognize his limitations, and will bow in reverent respect before some things greater than himself.

NOTE.—Some teachers may be interested to see how the preceding discussion would tend to elaborate in recitation the demonstrations as they appear in our best textbooks. It has seemed wise, accordingly, to add such a detailed demonstration. The particular theorem is selected because it offers illustrations of more points than does any other in the early part of the work.

The syllogistic method has been strictly adhered to until the last few steps of the concluding argument are reached. At this point the mind works so rapidly as to become impatient of talking out in slow words what it grasps instantly, if the preceding argument has been properly built up. You have seen children stand dominoes in a row at regular intervals, working with painstaking care to get them rightly placed. A single touch upon the last one overthrows it, and the others inevitably fall in rapid succession. So in this argument. When the conclusion that *CEC* cannot be a straight line is reached, all the rest of the structure comes tumbling down so rapidly that no time is left to do more than to watch each domino as it falls. See to it that the arguments are rightly placed as the demonstration is built up, and the rest will take care of itself.

Of course the syllogisms can be easily repeated in this last part as well, and it is often a good exercise to allow a class to supply them.

Book I. Proposition VI. Theorem.*

From a given point without a straight line a perpendicular can be drawn to the line, and but one.

Given the line AB and the point C without the line.

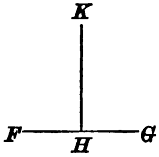
To prove (1) that from the point C one $\perp$ can be drawn to the line AB;

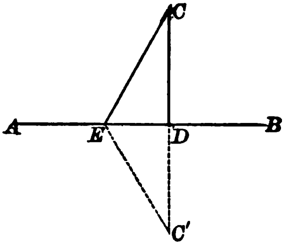
(2) that only one $\perp$ can be drawn from the point C to the line AB.

Proof. 1. Draw the auxiliary line FG, and let HK be drawn from $H \perp$ to AB.

2. At a given point in a straight line a perpendicular to the line can be drawn, and but one. (25)

$\therefore$ 3. HK is $\perp$ to FG at H.





Apply the line FG to the line AB, and move it along until HK passes through C. Let D be the point where H falls. Draw the line CD.

1. CD has two points, C and D, which coincide with points in HK by construction.

---

* From Wells's *Essentials of Plane and Solid Geometry.*

2. But one straight line can be drawn between two points. (Ax. 3.)

∴ 3. CD coincides with HK, and CDB coincides with and is equal to KHG.

1. KHG is a rt. ∠.

2. $\angle CDB = \angle KHG$.

∴ 3. CDB is a rt. ∠.

1. CDB is a rt. ∠.

2. When a line makes a right angle with another line, it is said to be perpendicular to it. (24)

∴ 3. CD is ⊥ to AB.

Hence one perpendicular can be drawn from the point C to the line AB.

If there can be another ⊥ from C to AB, let it be CE.
Produce CD, making $C'D = CD$. Draw $C'E$.

1. $C'D = CD$ by construction.

ED is ⊥ to CC' by construction.

2. If lines be drawn to the extremities of a straight line from any point in the perpendicular erected at its middle point, they make equal angles with the perpendicular. (44)

∴ 3. $\angle C'ED = \angle CED$.

1. $\angle C'ED = \angle CED$.

2. $\angle CED$ is a rt. ∠ by hypothesis.

∴ 3. $\angle C'ED$ is also a rt. ∠.

Add $\angle CED$ and $\angle C'ED$.

1. $\angle CED$ is a rt. $\angle$ by hypothesis.
$\angle C'ED$ is a rt. $\angle$ by proof.

2. If the sum of two adjacent angles is equal to two right angles, their exterior sides lie in the same straight line. (37)

$\therefore$ 3. CEC' is a straight line.

1. CEC' is a straight line by proof.
CDC' is a straight line by construction.

2. But one straight line can be drawn between two points. (Ax. 3.)

$\therefore$ 3. As we *know* that CDC' is a straight line by construction, CEC' cannot be a straight line.

If CEC' is not a straight line, then $CED + C'ED$ is not $=$ to two rt. $\angle$s.

If $CED + C'ED$ is not equal to two rt. $\angle$s, then CED, which is half of this sum, is not a rt. $\angle$.

If CED is not a rt. $\angle$, then CE is not $\perp$ to AB.

As CE is any other possible $\perp$ from C to AB, then CD is the only $\perp$ from C to the line AB.

Hence there can be only one perpendicular from C to the line AB.

3 2044 018 869 289

should
n or
ow.

THE BORROWER WILL BE CHARGED AN OVERDUE FEE IF THIS BOOK IS NOT RETURNED TO THE LIBRARY ON OR BEFORE THE LAST DATE STAMPED BELOW. NON-RECEIPT OF OVERDUE NOTICES DOES NOT EXEMPT THE BORROWER FROM OVERDUE FEES.

WIDENER
SEP 1 1 1995
BOOK DUE

WIDENER
MAY 1 6 1995
CANCELLED

WIDENER
FEB 1 0 2003
BOOK DUE

WIDENER
SEP 1 4 2003
CANCELLED